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THE PERTURBED RENEWAL RISK MODEL WITH STOCHASTIC INCOME

HU Yan, XIAO He-lu, YUE Sheng-jie, DENG Ying-chun

(*a. College of Mathematics and Computer Science;*
b. Key Laboratory of High Performance Computing and Stochastic Information Processing,
Ministry of Education of China,
Hunan Normal University, Changsha 410081, China)

Abstract: In this paper, the ruin problems under stochastic income in the perturbed renewal risk model are considered. By using Laplace transform and Lagrange interpolation formula, the asymptotic results for Gerber-Shiu function are derived when the individual stochastic premium sizes are exponentially distributed, which generalize the results of [4].

Keywords: Gerber-Shiu discounted penalty function; Laplace transform; stochastic income
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1 Introduction and Background

Risk processes involving stochastic income received a great deal of attention in recent years. In this kinds of models, the upward jumps can be interpreted as random income of an insurance company, while the downward jumps are interpreted as random losses of company. We remark that notable recent work allowing for such relaxation includes Labbé and Sendova [1], Zhao and Yin [4], Bao [5], as well as the references therein. Along this research line, it motivates us to study the perturbed renewal risk model with stochastic income.

The renewal risk model with stochastic income perturbed by a Brownian motion is given by

$$U(t) = u + ct + \sum_{i=1}^{M(t)} X_i - \sum_{j=1}^{N(t)} Y_j + \sigma B(t), \tag{1.1}$$

where $u > 0$ is the initial surplus, $c > 0$ is the constant premium rate. the process $\{\sum_{i=1}^{M(t)} X_i, t \geq 0\}$ is a compound Poisson process. The counting process $\{M(t), t \geq 0\}$ is homogenous Poisson process with parameter μ, the premium sizes $\{X_i, i \geq 1\}$ are independent and identically distributed (i.i.d.) and the probability density is $g(x) = \beta e^{-\beta x}, x \geq 0$.

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Biography: Hu Yan(1987–), female, born at Hengyang, Hunan, master, major in insurance mathematics. E–mail:huyanhnsd@163.com.

The counting process $N(t), t \geq 0$ representing the number of claims to the time t and $N(t)$ is a renewal process with i.i.d. inter-claim times $V_i, i \geq 1$. $\sigma > 0$ is the diffusion volatility and $B(t)$ is a standard Brownian motion staring from zero. In this paper, we assume that V_i is the generalized Erlang(n) distributed with n possibly different parameters $\lambda_1, \lambda_2, \cdots, \lambda_n$. The claim sizes $\{Y_j, j \geq 1\}$ are i.i.d. positive random variable with distribution function $F(y)$ and the density function $f(y)$.

Moreover, $\{M(t), t \geq 0\}$, $\{N(t), t \geq 0\}$, $\{X_i, i \geq 1\}$, $\{Y_j, j \geq 1\}$ and $\{B(t)\}_{t\geq}$ are assumed to be mutually independent. To guarantee that ruin is possible, we assume that following net profit condition holds:

$$c + \frac{\mu}{\beta} > \frac{E[Y_1]}{\sum\limits_{j=1}^{n} \frac{1}{\lambda_j}}. \tag{1.2}$$

Define the time of ruin

$$T = \inf\{t \geq 0,\ U(t) \leq 0\} \tag{1.3}$$

with $T = \infty$ if ruin never occur. Let $\delta \geq 0$ is a constant, and $\omega :\ [0, \infty) \times [0, \infty) \to [0, \infty)$ be a bounded function. Then, the Gerber-Shiu can be defined as:

$$m(u) = E[e^{-\delta T} \omega(U(T-), |U(T)|) I(T < \infty) | U(0) = u], \tag{1.4}$$

where $I(A)$ is an indicator function of event A. The quantity $\omega(U(T-), |U(T)|)$ is viewed as a penalty at the ruin time for the surplus immediately before ruin $U(T-)$ and the deficit at ruin $|U(T)|$. By observing the sample paths of $U(t)$, we known that ruin can be caused either by the oscillation of the Brownian motion or by a downward jump. We can decompose the Gerber-shiu function as

$$m(u) = m_s(u) + m_d(u), \tag{1.5}$$

where

$$m_s(u) = E[e^{-\delta T} \omega(U(T-), |U(T)|) I(T < \infty) | U(0) = u] \tag{1.6}$$

is the Gerber-Shiu function at ruin that is caused by a downward jump, and

$$\begin{aligned} m_d(u) &= E[e^{-\delta T} \omega(U(T-), |U(T)|) I(T < \infty) | U(0) = u] \\ &= \omega(0, 0) E[e^{-\delta T} I(T < \infty, U(T) = 0) | U(0) = u] \end{aligned} \tag{1.7}$$

is the Gerber-Shiu function at ruin that is caused by oscillation. In this paper, we set $\omega(0, 0) = 1$, then the following initial conditions hold

$$m(0) = m_d(0) = 1,\ \ m_s(0) = 0.$$

2 Laplace Transform

In this section, we derive the Laplace transform of the Gerber-Shiu function. First, we introduce the Dickson-Hipp operator T_s on a real-valued function $\varphi[0,\infty) \to R$ satisfying $\int_0^\infty |\varphi(x)|dx < \infty$, where s is a nonnegative real number. The operator T_s can be defined as

$$T_s\varphi(x) = \int_x^\infty e^{-s(y-x)}\varphi(y)dy,\ x \geq 0. \tag{2.1}$$

It's easy to see that the Laplace transform of $\varphi(s)$ can be expressed as $T_s\tilde{\varphi}(0) = \varphi(s)$, the operator T_s is commutative, and furthermore

$$T_sT_r\varphi(x) = T_rT_s\varphi(x) = \frac{T_s\varphi(x) - T_r\varphi(x)}{r-s},\ r \neq s. \tag{2.2}$$

For more properties of T_s, we refer to Dickson and Hipp [2] and Garrido [3].

Similar to the method are used in Zhao and Yin [4], we define the following modified claim process $N_j(t)$ of $N(t)$. Let $V_1^j = V_{1,j+1}, + \cdots + V_{1,n+1}, j = 0, 1, \cdots, n-1$ be the time until the first claim occurs while the other inter-claim times are as the same as that in $N(t)$. The only change is to replace $N(t)$ by $N_j(t)$ base on model (1.1). We define the modified model by $U_j(t)$ with $U_0(t) = U(t)$, and define the corresponding Gerber-Shiu function by $m_j(u)$, $j = 0, 1, \cdots, n-2$.

Now we consider $m_{s,j}(u)$, considering an infinitesimal time interval $(0,t)$, for $j = 0, 1, 2, \cdots, n-2$, we can get

$$\begin{aligned} m_{s,j}(u) &= (1-\mu t)(1-\lambda_{j+1}t)e^{-\delta t}m_{s,j}(u+ct+\sigma B(t)) \\ &\quad +\mu t(1-\lambda_{j+1})e^{-\delta t}\int_0^\infty m_{s,j}(u+ct+\sigma B(t)+x)g(x)dx \\ &\quad +(1-\mu t)(\lambda_{j+1}t)m_{s,j+1}(u+ct+\sigma B(t)) + o(t). \end{aligned}$$

By Taylor's expansion, dividing both sides by t, and letting $t \to 0$, the following equation can be derived

$$\begin{aligned} &(\lambda_{j+1}+\mu+\delta)m_{s,j}(u) - \frac{1}{2}\sigma^2 m''_{s,j}(u) - cm'_{s,j}(u) \\ = \ &\mu\int_0^\infty m_{s,j+1}(u+x)g(x)dx + \lambda_{j+1}m_{s,j+1}(u). \end{aligned} \tag{2.3}$$

Noting that $g(x) = \beta e^{-\beta x}$, so we have $\int_0^\infty m_{s,j}(u+x)g(x)dx = \beta T_\beta m_{s,j}(u)$. Hence, equation (2.3) can be rewritten as

$$\begin{aligned} &(\lambda_{j+1}+\mu+\delta)m_{s,j}(u) - \frac{1}{2}\sigma^2 m''_{s,j}(u) - cm'_{s,j}(u) \\ = \ &\mu\beta T_\beta m_{s,j}(u) + \lambda_{j+1}m_{s,j+1}(u). \end{aligned} \tag{2.4}$$

Similarly, we can obtain

$$\begin{aligned} m_{s,n-1}(u) &= (1-\mu t)(1-\lambda_n t)e^{-\delta dt} m_{s,n-1}(u+ct+\sigma B(t)) \\ &\quad +\mu(1-\lambda_n t)e^{-\delta t}\int_0^\infty m_{s,n-1}(u+ct+\sigma B(t)+x)g(x) \\ &\quad +(1-\mu t)(\lambda_n t)e^{-\delta t}[\int_0^u m_{s,0}(u+ct+\sigma B(t)-y)f(y)dy \\ &\quad +\int_{u+ct+\sigma B(t)}^\infty \omega(u+ct+\sigma B(t), y-u-ct-\sigma B(t))f(y)dy]+o(t). \end{aligned}$$

Which leads to

$$\begin{aligned} &(\lambda_n+\mu+\delta)m_{s,n-1}(u)-\frac{1}{2}\sigma^2 m''_{s,n-1}(u)-cm'_{s,n-1}(u) \\ = \ &\lambda_n[\int_0^u m_{s,0}(u-y)f(y)dy+\xi(u)]+\mu\beta T_\beta m_{s,n-1}(u), \end{aligned} \tag{2.5}$$

where $\xi(u)=\int_u^\infty \omega(u,y-u)f(y)dy$.

Applying the Laplace transform to both sides of equations (2.4) and (2.5), we have

$$\begin{aligned} &[(\lambda_{j+1}+\mu+\delta-\frac{1}{2}\sigma^2 s^2-cs)(\beta-s)-\mu\beta]\tilde{m}_{s,j}(s)-\lambda_{j+1}(\beta+s)\tilde{m}_{s,j+1}(s) \\ = \ &-\mu\beta\tilde{m}_{s,j}(\beta)-\frac{1}{2}\sigma^2(\beta-s)m'_{s,j}(0),\ \ j=0,1,\cdots,n-2 \end{aligned}$$

and

$$\begin{aligned} &[(\lambda_n+\mu+\delta-\frac{1}{2}\sigma^2 s^2-cs)(\beta-s)-\mu\beta]\tilde{m}_{s,n-1}(s)-\lambda_n(\beta-s)\tilde{f}(s)\tilde{m}_{s,0}(s) \\ = \ &-\mu\beta\tilde{m}_{s,n-1}(\beta)-\frac{1}{2}\sigma^2(\beta-s)m'_{s,n-1}(0)+\lambda_n(\beta-s)\tilde{\xi}(s). \end{aligned}$$

Let

$$\begin{cases} \tilde{\mathbf{m}}_s(s)=(\tilde{m}_{s,0}(s),\tilde{m}_{s,1}(s),\cdots,\tilde{m}_{s,n-1}(s))^T, \\ \boldsymbol{\theta}_1=\mu\beta(\tilde{m}_{s,0}(\beta),\tilde{m}_{s,1}(\beta),\cdots,\tilde{m}_{s,n-1}(\beta))^T, \\ \boldsymbol{\theta}_2=(\tilde{m}'_{s,0}(0),\tilde{m}'_{s,1}(0),\cdots,\tilde{m}'_{s,n-1}(0))^T, \\ \mathbf{e}=(0,\cdots,0,1)^T. \end{cases} \tag{2.6}$$

Then we can rewrite the above equations in the following matrix form

$$\mathbf{A}(s)\tilde{\mathbf{m}}_s(s)=-\boldsymbol{\theta}_1-\frac{1}{2}\sigma^2(\beta-s)\boldsymbol{\theta}_2+\lambda_n(\beta-s)\tilde{\xi}(s)\mathbf{e}, \tag{2.7}$$

where

$$\mathbf{A}(s)=\mathbf{B}(s)+\boldsymbol{\Lambda}(s),$$

while

$$\mathbf{B}(s)=(\beta-s)\begin{pmatrix} 0 & -\lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & -\lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda_{n-1} \\ -\lambda_n\tilde{f}(s) & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{2.8}$$

and

$$\mathbf{\Lambda}(s)=\mathrm{diag}((\lambda_1+\mu+\delta-\frac{1}{2}\sigma^2\mathrm{s}^2-\mathrm{cs})(\beta-\mathrm{s})-\mu\beta,\cdots,(\lambda_\mathrm{n}+\mu+\delta-\frac{1}{2}\sigma^2-\mathrm{cs})(\beta-\mathrm{s})-\mu\beta).$$

Similarly, for $m_{d,j}(u)$ we can obtain

$$\begin{aligned}&[(\lambda_{j+1}+\mu+\delta-\frac{1}{2}\sigma^2s^2-cs)(\beta-s)-\mu\beta]\tilde{m}_{d,j}(s)-\lambda_{j+1}(\beta+s)\tilde{m}_{s,j+1}(s)\\ =\ &-\mu\beta\tilde{m}_{d,j}(\beta)-\frac{1}{2}\sigma^2(\beta-s)m'_{d,j}(0)-(\beta-s)(\frac{\sigma^2}{2}s+c),\ \ j=0,1,\cdots,n-2\end{aligned}$$

and

$$\begin{aligned}&[(\lambda_n+\mu+\delta-\frac{1}{2}\sigma^2s^2-cs)(\beta-s)-\mu\beta]\tilde{m}_{d,n-1}(s)-\lambda_n(\beta-s)\tilde{f}(s)\tilde{m}_{d,0}(s)\\ =\ &-\mu\beta\tilde{m}_{d,n-1}(\beta)-\frac{1}{2}\sigma^2(\beta-s)m'_{d,n-1}(0)-(\beta-s)(\frac{\sigma^2}{2}s+c),\end{aligned}$$

let

$$\begin{cases}\tilde{\mathbf{m}}_d(s)=(\tilde{m}_{d,0}(s),\tilde{m}_{d,1}(s),\cdots,\tilde{m}_{d,n-1}(s))^T,\\ \boldsymbol{\theta}'_1=\mu\beta(\tilde{m}_{d,0}(\beta),\tilde{m}_{d,1}(\beta),\cdots,\tilde{m}_{d,n-1}(\beta))^T,\\ \boldsymbol{\theta}'_2=(\tilde{m}'_{d,0}(0),\tilde{m}'_{d,1}(0),\cdots,\tilde{m}'_{d,n-1}(0))^T,\\ \mathbf{e}'=(1,\cdots,1,1)^T.\end{cases}\tag{2.9}$$

Then, we can get the matrix form

$$\mathbf{A}(s)\tilde{\mathbf{m}}_d(s)=-\boldsymbol{\theta'}_1-\frac{1}{2}\sigma^2(\beta-s)\boldsymbol{\theta'}_2-(\beta-s)(\frac{\sigma^2}{2}s+c)\mathbf{e}'\tag{2.10}$$

and

$$\mathbf{A}(s)=\mathbf{B}(s)+\mathbf{\Lambda}(s),$$

where $\mathbf{B}(s)$ and $\mathbf{\Lambda}(s)$ are the same as the former results.

The solution of matrix equation (2.7) and (2.10) heavily depends on the roots of the equation $|\mathbf{A}(s)|=0$ which is equivalent to

$$\prod_{i=1}^n[(\lambda_i+\mu+\delta-\frac{1}{2}\sigma^2s^2-cs)(\beta-s)-\mu\beta]-[\prod_{i=1}^n\lambda_i](\beta-s)^n\tilde{f}(s)=0,\ s\in C.\tag{2.11}$$

We call (2.11) is the generalized Lundberg equation for the risk model (1.1).

Theorem 2.1 If $\delta>0$, $c>0$ the generalized Lundberg equation (2.11) has exactly $2n$ roots on the right half complex plane.

Proof Let $\mathbf{A}(s,v)=\mathbf{\Lambda}(s)+v\mathbf{B}(s)$, then $\mathbf{A}(s,1)=\mathbf{A}(s)$, for $\lambda_i+\mu+\delta-cs-\frac{1}{2}\sigma^2s^2=0$, then, we can get two roots of the equation

$$r_{1,2}=\frac{-c\pm\sqrt{c^2+2\sigma^2(\lambda_i+\mu+\delta)}}{\sigma^2}.\tag{2.12}$$

Obviously, r_1 is positive root, ε denote a circle with center at $(R, 0)$ and radius R, where $R = \frac{-c+\sqrt{c^2+2\sigma^2(\lambda_i+\mu+\delta)}}{\sigma^2}$, let $\{s : \ |s-R| \geq R, \ \mathcal{R}(s) \geq 0\}$ by $\bar{\varepsilon}$, we first prove that, for $0 \leq v \leq 1$, $\{|A(s,v)| \neq 0, \quad s \in \bar{\varepsilon}\}$. If $0 \leq v \leq 1$, $s \in \bar{\varepsilon}$, then

$$
\begin{aligned}
& |\lambda_i + \mu + \delta - \frac{1}{2}\sigma^2 s^2 - cs - \frac{\mu\beta}{\beta - s}| \\
\geq \ & |\lambda_i + \mu + \delta - \frac{1}{2}\sigma^2 s^2 - cs| - \frac{\mu\beta}{|\beta - s|} \\
= \ & |\frac{1}{2}\sigma^2 s^2 - \frac{1}{2}\sigma^2 R^2 + cs - cR + \frac{1}{2}\sigma^2 R^2 + cR - \lambda_i - \mu - \delta| - \frac{\mu\beta}{|\beta - s|} \\
\geq \ & |s - R||\frac{1}{2}\sigma^2 s + \frac{1}{2}\sigma^2 R + c| - |\frac{1}{2}\sigma^2 R + cR - \lambda_i - \mu - \delta| - \frac{\mu\beta}{|\beta - s|} \\
\geq \ & R|\frac{1}{2}\sigma^2 s + \frac{1}{2}\sigma^2 R + c| - |\frac{1}{2}\sigma^2 R + cR - \lambda_i - \mu - \delta| - \frac{\mu\beta}{|\beta - s|} \\
\geq \ & \frac{1}{2}\sigma^2 R^2 + cR - \frac{1}{2}\sigma^2 R^2 - cR + \lambda_i + \mu + \delta - \frac{\mu\beta}{|\beta - s|} \\
= \ & \lambda_i + \mu + \delta - \frac{\mu\beta}{|\beta - s|} \\
\geq \ & \lambda_i + \mu + \delta - \mu\beta\frac{1}{\beta} \geq v\lambda_1, \quad i = 1, 2, \cdots, n-1.
\end{aligned}
$$

So, we can derive

$$
|(\lambda_i + \mu + \delta - \frac{1}{2}\sigma^2 s^2 - cs)(\beta - s) - \mu\beta| > v\lambda_i|\beta - s|.
$$

Similarly,

$$
|(\lambda_n + \mu + \delta - \frac{1}{2}\sigma^2 s^2 - cs)(\beta - s) - \mu\beta| > \lambda_n|\beta - s| \geq v\lambda_n|\beta - s|\tilde{f}(0) \geq \lambda_n|(\beta - s)\tilde{f}(s)|.
$$

By Rouchés theorem, it's easy to know that equation (2.11) have $2n$ roots in ε.

If we denote the root with the smallest real part by $\rho_{2n}(\delta)$, then it is easy to see that $\rho_{2n}(\delta) \to 0$ as $\delta \to 0$. The roots of generalized Lundberg equation play an important role in the rest of this paper. In what follow, we denote them by $\rho_1, \rho_2, \cdots, \rho_{2n}$ for simplicity, and only consider the case when they are all distinct.

Theorem 2.2 Assuming the roots of the generalized Lundberg equation (2.11) are distinct, then the Laplace transform of Gerber-Shiu function can be expressed as

$$
\tilde{m}_s(s) = \frac{-\mathbf{A}_1^*(s)[\boldsymbol{\theta}_1 + \frac{1}{2}\sigma^2(\beta - s)\boldsymbol{\theta}_2] + (\prod_{i=1}^n \lambda_i)(\beta - s)^n \tilde{\xi}(s)}{|\mathbf{A}(s)|}, \tag{2.13}
$$

$$
\tilde{m}_d(s) = \frac{-\mathbf{A}_1^*(s)[\boldsymbol{\theta}'_1 + \frac{1}{2}\sigma^2(\beta - s)\boldsymbol{\theta}'_2 + (\beta - s)(\frac{\sigma^2}{2}s + c)\mathbf{e}']}{|\mathbf{A}(s)|}, \tag{2.14}
$$

where

$$\begin{aligned}\mathbf{A}_1^*(s) &= (\prod_{j=2}^{n}[(\lambda_j+\mu+\delta-\frac{1}{2}\sigma^2s^2-cs)(\beta-s)-\mu\beta],\\ &\quad \lambda_1(\beta-s)\prod_{j=3}^{n}[(\lambda_j+\mu+\delta-\frac{1}{2}\sigma^2s^2-cs)(\beta-s)-\mu\beta],\cdots,\\ &\quad \prod_{i=1}^{n-2}[(\lambda_i(\beta-s)][\lambda_n+\mu+\delta-\frac{1}{2}\sigma^2s^2-cs)(\beta-s)-\mu\beta],\prod_{i=1}^{n-1}[(\lambda_i(\beta-s)])\end{aligned}$$

is the first row of the matrix $\mathbf{A}^*(s)$.

Proof Let $\mathbf{A}^*(s)$ denote the adjoint matrix of $\mathbf{A}(s)$, by equation (2.7), we can obtain

$$\tilde{\mathbf{m}}_s(s)=\frac{\mathbf{A}^*(s)[-\boldsymbol{\theta}_1-\frac{1}{2}\sigma^2(\beta-s)\boldsymbol{\theta}_2+\lambda_n(\beta-s)\tilde{\xi}(s)\mathbf{e}]}{|\mathbf{A}(s)|},$$

then we can get

$$\begin{aligned}\tilde{m}_s(s)=\tilde{m}_{s,0}(s) &= \frac{\mathbf{A}_1^*(s)[-\boldsymbol{\theta}_1-\frac{1}{2}\sigma^2(\beta-s)\boldsymbol{\theta}_2+\lambda_n(\beta-s)\tilde{\xi}(s)\mathbf{e}]}{|\mathbf{A}(s)|}\\ &= \frac{-\mathbf{A}_1^*(s)[\boldsymbol{\theta}_1+\frac{1}{2}\sigma^2(\beta-s)\boldsymbol{\theta}_2]+(\prod_{i=1}^{n}\lambda_i)(\beta-s)^n\tilde{\xi}(s)}{|\mathbf{A}(s)|}.\end{aligned}$$

Similarly, equation (2.14) hold.

3 Asymptotic Results for Laplace Transform of Gerber-Shiu Function

It's easy to check that the Laplace transform of gerber-shiu function are dependent on $\boldsymbol{\theta}_1$, $\boldsymbol{\theta}_2$, $\boldsymbol{\theta}'_1$ and $\boldsymbol{\theta}'_2$. The main goal of this section is to derive the Asymptotic results for Laplace transform of Gerber-Shiu function by the Lagrange interpolation formula.

Theorem 3.1 The asymptotic results for Laplace transform of Gerber-Shiu function can be expressed as

$$\tilde{m}_s(s)\approx\frac{(\prod\limits_{i=1}^{n}\lambda_i)[(\beta-s)^n\tilde{\xi}(s)-\sum\limits_{i=1}^{2n}\frac{(\beta-\rho_j)^n\tilde{\xi}(\rho_j)}{\pi_{2n,j}(\rho_j)}\pi_{2n,j}(s)]}{\prod\limits_{i=1}^{n}[(\lambda_i+\mu+\delta-\frac{1}{2}\sigma^2s^2-cs)(\beta-s)-\mu\beta]-[\prod\limits_{i=1}^{n}\lambda_i](\beta-s)^n\tilde{f}(s)} \tag{3.1}$$

and

$$\tilde{m}_d(s)\approx\frac{\prod_{i=1}^{n-1}(\lambda_i)[\sum\limits_{j=1}^{2n}\frac{(\beta-\rho_j)^n(\frac{\sigma^2}{2}\rho_j+c)}{\pi_{2n,j}(\rho_j)}\pi_{2n,j}(s)-(\beta-s)^n(\frac{\sigma^2}{2}s+c)]}{\prod\limits_{i=1}^{n}[(\lambda_i+\mu+\delta-\frac{1}{2}\sigma^2s^2-cs)(\beta-s)-\mu\beta]-[\prod\limits_{i=1}^{n}\lambda_i](\beta-s)^n\tilde{f}(s)}, \tag{3.2}$$

where

$$\pi_{2nj}(s)=\begin{cases}1,\text{if } \ 2n=j=1;\\ \prod_{l=1,l\neq j}^{2n}(s-\rho_l), \quad \text{if } \ j=1,2,\cdots,2n.\end{cases}$$

Proof By equation (2.13), we known that $\mathbf{A}_1^*(s)[\boldsymbol{\theta}_1+\frac{1}{2}\sigma^2(\beta-s)\boldsymbol{\theta}_2]$ is a polynomial function on s of degree $3n-2$. Since $|\mathbf{A}(\rho_j)|=0$, for $j=1,2,\cdots,2n$, we can derive

$$\mathbf{A}_1^*(\rho_j)[\boldsymbol{\theta}_1+\frac{1}{2}\sigma^2(\beta-\rho_j)\boldsymbol{\theta}_2]=(\prod_{i=1}^{n}\lambda_i)(\beta-\rho_j)^n\tilde{\xi}(\rho_j),\ \ j=1,2,\cdots,2n,$$

by using Lagrange interpolation formula, we obtain

$$\mathbf{A}_1^*(s)[\boldsymbol{\theta}_1+\frac{1}{2}\sigma^2(\beta-s)\boldsymbol{\theta}_2]\approx(\prod_{i=1}^{n}\lambda_i)\sum_{i=1}^{2n}\frac{(\beta-\rho)^n\tilde{\xi}(\rho_j)}{\pi_{2n,j}(\rho_j)}\pi_{2n,j}(s). \tag{3.3}$$

Substituting (3.3) into (2.13) we can get the following euqation

$$\tilde{m}_s(s)\approx\frac{(\prod_{i=1}^{n}\lambda_i)\sum_{i=1}^{2n}\frac{(\beta-\rho)^n\tilde{\xi}(\rho_j)}{\pi_{2n,j}(\rho_j)}\pi_{2n,j}(s)-(\prod_{i=1}^{n}\lambda_i)(\beta-s)^n\tilde{\xi}(s)}{|\mathbf{A}(s)|}. \tag{3.4}$$

Similarly, result (3.2) can be derived.

4 Asymptotic Solutions for Rational Family Claim-Size Distribution

In this section, we consider the case where the claim amount distribution $F(u)$ belongs to the rational family, that is, its density Laplace transform is of the form

$$\tilde{f}(s)=\frac{l_{m-1}(s)}{l_m(s)},\ \ m\in N^+, \tag{4.1}$$

where $l_m(s)$ is a polynomial of degrees m, $l_m(s)=0$ which only have roots with negative real parts, $l_{m-1}(s)$ is a polynomial of degrees $m-1$ or less, all have leading coefficient 1 and satisfy $l_m(0)=l_{m-1}(0)$.

$$\begin{aligned}\tilde{m}_s(s)&\approx\frac{(\prod\limits_{i=1}^{n}\lambda_i)[(\beta-s)^n\tilde{\xi}(s)-\sum\limits_{i=1}^{2n}\frac{(\beta-\rho_j)^n\tilde{\xi}(\rho_j)}{\pi_{2n,j}(\rho_j)}\pi_{2n,j}(s)]}{\prod\limits_{i=1}^{n}[(\lambda_i+\mu+\delta-\frac{1}{2}\sigma^2s^2-cs)(\beta-s)-\mu\beta]-[\prod\limits_{i=1}^{n}\lambda_i](\beta-s)^n\tilde{f}(s)}\\&=\frac{l_m(s)(\prod\limits_{i=1}^{n}\lambda_i)[((\beta-s)^n\tilde{\xi}(s)-\sum\limits_{i=1}^{2n}\frac{(\beta-\rho_j)^n\tilde{\xi}(\rho_j)}{\pi_{2n,j}(\rho_j)}\pi_{2n,j}(s)]}{l_m(s)\prod\limits_{i=1}^{n}[(\lambda_i+\mu+\delta-\frac{1}{2}\sigma^2s^2-cs)(\beta-s)-\mu\beta]-[\prod\limits_{i=1}^{n}\lambda_i](\beta-s)^nl_{m-1}(s)}.\end{aligned} \tag{4.2}$$

Obviously, $l_m(s)|\mathbf{A}(s)|$ is a polynomial of degree $2n+m+1$, with leading coefficient $(\sigma^2/2)^n$, the equation $l_m(s)|\mathbf{A}(s)|=0$ has exactly $2n+m+1$ roots, then $l_m(s)|\mathbf{A}(s)|$ can be expressed as

$$l_m(s)|\mathbf{A}(s)|=(\frac{\sigma^2}{2})^n\prod_{j=1}^{2n}(s-\rho_j)\prod_{k=1}^{m+1}(s+R_k). \tag{4.3}$$

We remark that all $-R_k,\ k=1,\cdots,m+1$ are the roots of $l_m(s)|\mathbf{A}(s)|=0$ which have a negative real part.

$$\begin{aligned}
&(\beta-s)^n\tilde{\xi}(s)-\sum_{i=1}^{2n}\frac{(\beta-\rho_j)^n\tilde{\xi}(\rho_j)}{\pi_{2n,j}(\rho_j)}\pi_{2n,j}(s)\\
=&\sum_{i=1}^{2n}\frac{\pi_{2n,j}(s)}{\pi_{2n,j}(\rho_j)}[(\beta-s)^n\tilde{\xi}(s)-(\beta-\rho_j)^n\tilde{\xi}(\rho_j)]\\
=&\sum_{i=1}^{2n}\frac{\pi_{2n,j}(s)}{\pi_{2n,j}(\rho_j)}[(\beta-s)^n\tilde{\xi}(s)-(\beta-\rho_j)^n\tilde{\xi}(s)+(\beta-\rho_j)^n\tilde{\xi}(s)-(\beta-\rho_j)^n\tilde{\xi}(\rho_j)]\\
=&-\prod_{i=1}^{2n}(s-\rho_j)[\sum_{j=1}^{2n}\frac{(\beta-\rho_j)^n}{\pi_{2n,j}(\rho_j)}T_sT_{\rho_j}\xi(0)+\sum_{j=1}^{2n}\frac{\tilde{\xi}(s)}{\pi_{2n,j}}\frac{(\beta-\rho_j)^n-(\beta-s)^n}{s-\rho_j}]\\
=&-\prod_{i=1}^{2n}(s-\rho_j)\sum_{j=1}^{2n}\frac{(\beta-\rho_j)^n}{\pi_{2n,j}(\rho_j)}T_sT_{\rho_j}\xi(0),
\end{aligned}\tag{4.4}$$

where the last equality is due to the following formula in interpolation theory

$$\sum_{j=1}^{m}\frac{(\rho_j-s)^k}{\prod_{i=1,i\neq j}^{m}(\rho_j-\rho_i)}=\begin{cases}1, & k=m-1,\\ 0, & k=0,1,\cdots,n-2.\end{cases}$$

Similarly, we have

$$\sum_{j=1}^{2n}\frac{(\beta-\rho_j)^n(\frac{\sigma^2}{2}\rho_j+c)}{\pi_{2n,j}(\rho_j)}\pi_{2n,j}(s)-(\beta-s)^n(\frac{\sigma^2}{2}s+c)=\frac{\sigma^2}{2}\prod_{i=1}^{2n}(s-\rho_i).\tag{4.5}$$

Substituting (4.4) into (4.2) we obtain

$$\tilde{m}_s(s)\approx(\prod_{i=1}^{n}\frac{2\lambda_i}{\sigma^2})(\frac{l_m(s)}{\prod_{k=1}^{m+1}(s+R_k)})[-\sum_{j=1}^{2n}\frac{(\beta-\rho_j)^n}{\pi_{2n,j}(\rho_j)}T_sT_{\rho_j}\xi(0)].\tag{4.6}$$

If all R'_j are distinct, performing partial fraction leads to

$$\frac{(\prod_{i=1}^{n}\frac{2\lambda_i}{\sigma^2})l_m(s)[-\frac{(\beta-\rho_j)^n}{\pi_{2n,j}(\rho_j)}]}{\prod_{k=1}^{m+1}(s+R_k)}=\sum_{k=1}^{m+1}\frac{A_{j,k}}{s+R_k},$$

where $A_{j,k}=\frac{(\prod_{i=1}^{n}\frac{2\lambda_i}{\sigma^2})l_m(-R_k)}{\prod_{l=1,l\neq k}^{m+1}(R_l-R_k)}[-\frac{(\beta-\rho_j)^n}{\pi_{2n,j}(\rho_j)}]$, then, we have

$$\tilde{m}_s(s)\approx\sum_{j=1}^{2n}\sum_{k=1}^{m+1}\frac{A_{j,k}}{s+R_k}T_sT_{\rho_j}\xi(0).\tag{4.7}$$

Similarly, the following result can be derived

$$m_d(s)\approx\sum_{k=1}^{m+1}\frac{B_k}{s+R_k},\tag{4.8}$$

where $B_k = \frac{\prod_{i=1}^{n-1}(\lambda_i) l_m(-R_k)}{\prod_{l=1,l\neq k}^{m+1}(R_l - R_k)}$. Finally, inverting the Laplace tranforms in the above formulas yields the following asymptotic expression for $m_s(u)$ and $m_d(u)$.

Theorem 4.1 The Gerber-Shiu function asymptotic results can be expressed as follows.

$$m_s(u) \approx \sum_{j=1}^{2n}\sum_{k=1}^{m+1} A_{k,j} e^{-R_k u} * T_{\rho_j}\xi(u), \tag{4.9}$$

$$m_d(u) \approx \sum_{k=1}^{m+1} B_k e^{-R_k u}, \tag{4.10}$$

where $*$ denote the convolution operator.

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具有随机收入的扰动更新风险模型

胡 燕, 肖和录, 乐胜杰, 邓迎春

(湖南师范大学a.数学与计算机科学学院; b. 高性能计算与随机信息处理省部共建教育部重点实验室,

湖南 长沙 410081)

摘 要: 本文研究了一类随机收入的扰动更新风险模型的破产问题. 运用拉普拉斯变换以及拉格朗日差值公式得到了Gerber-Shiu函数的拉普拉斯变换的渐近表达式, 推广了文献[4]中的结论.

关 键 词: Gerber-Shiu 折罚函数; 拉普拉斯变换; 随机收入

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