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THE LAW OF SINES FOR AN n-SIMPLEX IN HYPERBOLIC SPACE AND SPHERICAL SPACE AND ITS APPLICATIONS

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Abstract: In this paper, the law of sines and related geometric inequalities for an n-simplex in an n-dimensional hyperbolic space and an n-dimensional spherical space are studied. By using the theory and method of distance geometry, we give the law of sines for an n-simplex in an n-dimensional hyperbolic space and an n-dimensional spherical space. As applications, we obtain Hadamard type inequalities and Veljan-Korchmaros type inequalities in n-dimensional hyperbolic space and n-dimensional spherical space. In addition, some new geometric inequality about "metric addition" involving volume and n-dimensional space angle of simplex in $H^n(K)$ and $S^n(K)$ is established.

Keywords: hyperbolic space; spherical space; the law of sine; metric addition; geometric inequality

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1 Introduction

The law of sine of triangle ($\triangle ABC$) in the Euclidean plane is well known as follows

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \frac{abc}{2S}, \tag{1.1}$$

where $S = \sqrt{p(p-a)(p-b)(p-c)}$, $p = \frac{1}{2}(a+b+c)$.

Let a, b, c be the edge-lengths of a triangle ABC in the hyperbolic space with curvature -1. Then we have the law of sine of hyperbolic triangle ABC as follows (see [1])

$$\frac{\sinh a}{\sin A} = \frac{\sinh b}{\sin B} = \frac{\sinh c}{\sin C} = \frac{\sinh a \sinh b \sinh c}{2\Delta}, \tag{1.2}$$

where $\Delta = \sqrt{\sinh p(\sinh p - \sinh a)(\sinh p - \sinh b)(\sinh p - \sinh c)}$, $p = \frac{1}{2}(a+b+c)$.

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We denote by a, b, c the edge-lengths of a triangle ABC in the spherical space with curvature 1. Then we have the law of sine of spherical triangle ABC as follows (see [2])

$$\frac{\sin a}{\sin A}=\frac{\sin b}{\sin B}=\frac{\sin c}{\sin C}=\frac{\sin a \sin b \sin c}{2\Delta}, \tag{1.3}$$

where $\Delta=\sqrt{\sin p(\sin p-\sin a)(\sin p-\sin b)(\sin p-\sin c)}$, $p=\frac{1}{2}(a+b+c)\in(0,\pi)$.

The law of sine of triangle in Euclidean plane were generalized to the n-dimensional simplex in n-dimensional Euclidean space E^n. Let $\{A_0, A_1, \cdots, A_n\}$ be the vertex sets of n-dimensional simplex $\Omega_n(E)$ in the n-dimensional Euclidean E^n. Denote by V the volume of the simplex $\Omega_n(E)$, and F_i $(i=0,1,\cdots,n)$ the areas of i-th face $f_i=\{A_0,\ A_1,\ \cdots,\ A_{i-1},\ A_{i+1},\ \cdots,\ A_n\}$ ($(n-1)$-dimensional simplex) of the simplex $\Omega_n(E)$. In 1968, Bators defined the n-dimensional sines of the n-dimensional vertex angles α_i $(i=0,1,\cdots,n)$ for the n-dimensional simplex $\Omega_n(E)$, and established the law of sines for $\Omega_n(E)$ as follows (see [3])

$$\frac{F_0}{\sin\alpha_0}=\frac{F_1}{\sin\alpha_1}=\cdots=\frac{(n-1)!\prod_{j=0}^{n}F_j}{(nV)^{n-1}}. \tag{1.4}$$

Obviously, formula (1.4) is generalization of formula (1.1) in n-dimensional Euclidean space E^n. Then, some different forms of generalization about formula (1.1) was given in [4, 5, 6].

From 1970s, many geometry researchers were attempted to generalize formulas (1.2) and (1.3) to an n-dimensional hyperbolic simplex (spherical simplex), to establish the law of sines in n-dimensional hyperbolic space H^n and in n-dimensional spherical space S^n. In 1978, Erikson defined the n-dimensional polar sine of i-th face $f_i(P_i\notin f_i)$ of $\Omega_n(S)$ in $S_{n,1}$ (see [7]) as follows

$$^{n}\mathrm{Pol}\sin F_i=|[\nu_0,\nu_1,\cdots,\nu_{i-1},\nu_{i+1},\cdots,\nu_n]| \quad (i=0,1,\cdots,n). \tag{1.5}$$

Let $^{n}\sin P_i$ be the n-dimensional sine of the i-th angle of $\Omega_n(S)$ (see [7]). The law of sines in the n-dimensional spherical space $S_{n,1}$ was obtained in [7] as follows

$$\frac{^{n}\mathrm{Pol}\sin F_0}{^{n}\sin P_0}=\frac{^{n}\mathrm{Pol}\sin F_1}{^{n}\sin P_1}=\cdots=\frac{^{n}\mathrm{Pol}\sin F_n}{^{n}\sin P_n}. \tag{1.6}$$

In 1980s, Yang and Zhang (see [8, 9, 10]) made a large number of basic works of geometric inequality in n-dimensional hyperbolic space H^n and in n-dimensional spherical space S^n, and established the law of cosines in n-dimensional hyperbolic space H^n and n-dimensional spherical space S^n. But they did not establish the law of sines in n-dimensional hyperbolic space H^n and n-dimensional spherical space S^n. In addition, some new geometric inequality about "metric addition" involving volume and n-dimensional space angle of simplex in $H^n(K)$ and $S^n(K)$ is established.

In this paper, we give generalizations of (1.2) and (1.3) in n-dimensional hyperbolic space H^n and in n-dimensional spherical space S^n, and establish the law of sines n-dimensional simplex in n-dimensional hyperbolic space H^n and n-dimensional spherical space S^n. As their applications, we obtain Veljan-Korchmaros type inequalities and Hadamard type inequalities in n-dimensional hyperbolic space H^n and n-dimensional spherical space S^n.

2 The Law of Sine in Hyperbolic Space

We consider the model of the hyperbolic space in Euclidean space (see [9]).

Let B be a set whose elements $x(x_1, x_2, \cdots, x_n)$ are in an n-dimensional vector space and meet the following condition $x_1^2 + x_2^2 + \cdots + x_n^2 < 1$. Given a distance between x and y in the set B, denote by xy satisfying

$$\cosh \sqrt{-K}xy = \frac{1 - x_1y_1 - x_2y_2 - \cdots - x_ny_n}{\sqrt{1 - x_1^2 - x_2^2 - \cdots - x_n^2}\sqrt{1 - y_1^2 - y_2^2 - \cdots - y_n^2}}.$$

Then the metric space with this distance in R^{n+1} is called n-dimensional hyperbolic space with curvature $K(<0)$, denote by $H^n(K)$.

Let $\Sigma_n(H)$ be an n-dimensional simplex in the n-dimensional hyperbolic space $H^n(K)$, and $\{P_0, P_1, \cdots, P_n\}$ be its vertexes, a_{ij} $(0 \leqslant i, j \leqslant n)$ be its edge-length, V be its volume, respectively.

To give the law of sines in n-dimensional hyperbolic space $H^n(K)$, we give the following definition.

Definition 2.1 Suppose that $\Sigma_n(H) = \{P_0, P_1, \cdots, P_n\}$ be an n-dimensional simplex in n-dimensional hyperbolic space $H^n(K)$. n edges P_0P_i $(i = 1, 2, \cdots, n)$ with initial point P_0 form an n-dimensional space angle P_0 of the simplex $\Sigma_n(H)$. Let $\widehat{i,j}$ be the angle formed by two edges P_0P_i and P_0P_j. The sine of the n-dimensional space angle P_0 of the simplex $\Sigma_n(H)$ is defined as follows

$$\sin P_0 = (\det Q_0)^{\frac{1}{2}}, \tag{2.1}$$

where

$$Q_0 = \begin{bmatrix} 1 & & & \cos\widehat{i,j} \\ & 1 & & \\ & & \ddots & \\ \cos\widehat{i,j} & & & 1 \end{bmatrix} \quad (i, j = 1, 2, \cdots, n).$$

Similarly, we can define the sine of the n-dimensional space angle P_i $(i = 1, 2, \cdots, n)$ of the simplex $\Sigma_n(H)$.

At first, we prove that this definition is sensible.

Actually, for n-ray P_0P_i $(i = 1, 2, \cdots, n)$ of the simplex $\Sigma_n(H)$ in n-dimensional hyperbolic space $H^n(K)$, and denote by $\widehat{i,j}$ $(i, j = 1, 2, \cdots, n)$ the included angle between two rays P_0P_i and P_0P_j. According to [12], we know that there exist n-ray $P_0'P_i'$ $(i = 1, 2, \cdots, n)$ which are independence in n-dimensional Euclidean space E^n, such that the included angle between two rays $P_0'P_i'$ and $P_0'P_j'$ is also $\widehat{i,j}$ $(i, j = 1, 2, \cdots, n)$. Assume that $\overrightarrow{\alpha_i}$ $(i = 1, 2, \cdots, n)$ denote the unit vector of the vector $\overrightarrow{P_0'P_i'}$ $(i, j = 1, 2, \cdots, n)$, then the unit vectors $\alpha_1, \alpha_2, \cdots, \alpha_n$

are also independence. So the Gram matrix Q_0 of the unit vectors $\alpha_1, \alpha_2, \cdots, \alpha_n$ is positive and it is easy to know that $0 < \det Q_0 \leqslant 1$. Therefore, this definition is sensible.

Remark Especially two-dimensional space angle of two-dimensional hyperbolic simplex (that is hyperbolic triangle) is just interior angle of hyperbolic triangle. So the n-dimensional space angle of n-dimensional simplex in n-dimensional hyperbolic space $H^n(K)$ is extension of interior angle of hyperbolic triangle.

Definition 2.2 (see [12]) Suppose that $\Sigma_n(H) = \{P_0, P_1, \cdots, P_n\}$ be an n-dimensional simplex in n-dimensional hyperbolic space $H^n(K)$, its volume V is the real number satisfying

$$\sinh^2 \sqrt{-K}V = \frac{(-1)^n}{2^n \cdot (n!)^2} \det(\Lambda_n(H)), \tag{2.2}$$

where $\Lambda_n(H) = (\cosh \sqrt{-K} a_{ij})_{i,j=0}^n$.

Theorem 2.1 Suppose that $\Sigma_n(H) = \{P_0, P_1, \cdots, P_n\}$ be an n-dimensional simplex in the n-dimensional hyperbolic space $H^n(K)$, we have

$$\begin{aligned} & \frac{\prod\limits_{1\leqslant i<j\leqslant n} \sinh \sqrt{-K} a_{ij}}{\sin P_0} = \frac{\prod\limits_{0\leqslant i<j\leqslant n, i,j\neq 1} \sinh \sqrt{-K} a_{ij}}{\sin P_1} = \cdots \\ = \ & \frac{\prod\limits_{0\leqslant i<j\leqslant n-1} \sinh \sqrt{-K} a_{ij}}{\sin P_n} = \frac{\prod\limits_{0\leqslant i<j\leqslant n} \sinh \sqrt{-K} a_{ij}}{2^{\frac{n}{2}} \cdot n! \cdot \sinh \sqrt{-K}V}, \end{aligned} \tag{2.3}$$

where definitions of $\sin P_i$ $(i = 0, 1, \cdots)$ are the same as Definition 2.1.

Remark When $n = 2$ in Theorem 2.1, it is the law of sine of a hyperbolic triangle.

Lemma 2.2 (see [1]) (the law of cosine of a hyperbolic triangle) For hyperbolic triangle ABC in $H^2(-1)$, then

$$\cosh a = \cosh b \cdot \cosh c - \sinh b \cdot \sinh c \cdot \cos A, \tag{2.4}$$

where a, b, c be edge-lengths of hyperbolic triangle ABC and A be the interior angle.

Lemma 2.3 Suppose that $\Sigma_n(H) = \{P_0, P_1, \cdots, P_n\}$ be an n-dimensional simplex in n-dimensional hyperbolic space $H^n(K)$, we have

$$\sinh^2 \sqrt{-K}V = \frac{1}{2^n \cdot (n!)^2} \Big(\prod_{i=1}^n \sinh^2 \sqrt{-K} a_{0i}\Big) \cdot \begin{vmatrix} 1 & & & \cos \widehat{i,j} \\ & 1 & & \\ & & \ddots & \\ \cos \widehat{i,j} & & & 1 \end{vmatrix}, \tag{2.5}$$

where $\widehat{i,j}$ $(i, j = 1, 2, \cdots, n)$ be the included angle between the edges P_0P_i and P_0P_j.

Proof Assume that the row or the column number of determinant $\det(\Lambda_n(H))$ begins from 0. Now, we transform the determinant $\det(\Lambda_n(H))$ as follows:

(1) for $i = 1, 2, \cdots, n$, plus $(-\cosh \sqrt{-K} a_{0i})$ times the 0-th row to the i-th row;

(2) expanding the determinant at the 0-th column;

(3) for $i=1,2,\cdots,n$, $\frac{1}{(\sinh\sqrt{-K}a_{0i})}$ times the i-th row and $\frac{1}{(\sinh\sqrt{-K}a_{0i})}$ times the i-th column.

$$\det(\Lambda_n(H))=\begin{vmatrix} 1 & \cosh\sqrt{-K}a_{01} & \cdots & \cosh\sqrt{-K}a_{0n} \\ \cosh\sqrt{-K}a_{10} & 1 & \cdots & \cosh\sqrt{-K}a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ \cosh\sqrt{-K}a_{n0} & \cosh\sqrt{-K}a_{n1} & \cdots & 1 \end{vmatrix}$$

$$=\begin{vmatrix} 1 & \cosh\sqrt{-K}a_{01} & \cdots & \cosh\sqrt{-K}a_{0n} \\ 0 & 1-\cosh^2\sqrt{-K}a_{01} & \cdots & \cosh\sqrt{-K}a_{1n}-\cosh\sqrt{-K}a_{0n}\cosh\sqrt{-K}a_{01} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cosh\sqrt{-K}a_{n1}-\cosh\sqrt{-K}a_{0n}\cosh\sqrt{-K}a_{01} & \cdots & 1-\cosh^2\sqrt{-K}a_{0n} \end{vmatrix}$$

$$=\begin{vmatrix} 1-\cosh^2\sqrt{-K}a_{01} & \cdots & \cosh\sqrt{-K}a_{1n}-\cosh\sqrt{-K}a_{0n}\cosh\sqrt{-K}a_{01} \\ \cdots & \cdots & \cdots \\ \cosh\sqrt{-K}a_{n1}-\cosh\sqrt{-K}a_{0n}\cosh\sqrt{-K}a_{01} & \cdots & 1-\cosh^2\sqrt{-K}a_{0n} \end{vmatrix}$$

$$=\Big(\prod_{i=1}^{n}\sinh^2\sqrt{-K}a_{0i}\Big)\begin{vmatrix} \frac{1-\cosh^2\sqrt{-K}a_{01}}{\sinh^2\sqrt{-K}a_{01}} & \cdots & \frac{\cosh\sqrt{-K}a_{1n}-\cosh\sqrt{-K}a_{0n}\cosh\sqrt{-K}a_{01}}{\sinh\sqrt{-K}a_{01}\sinh\sqrt{-K}a_{0n}} \\ \cdots & \cdots & \cdots \\ \frac{\cosh\sqrt{-K}a_{n1}-\cosh\sqrt{-K}a_{0n}\cosh\sqrt{-K}a_{01}}{\sinh\sqrt{-K}a_{01}\sinh\sqrt{-K}a_{0n}} & \cdots & \frac{1-\cosh^2\sqrt{-K}a_{0n}}{\sinh^2\sqrt{-K}a_{0n}} \end{vmatrix}$$

$$=(-1)^n\Big(\prod_{i=1}^{n}\sinh^2\sqrt{-K}a_{0i}\Big)\begin{vmatrix} 1 & & \cos\widehat{i,j} \\ & 1 & \\ & & \ddots & \\ \cos\widehat{i,j} & & & 1 \end{vmatrix}.$$

By (2.3), we have

$$\det(\Lambda_n(H))=\frac{2^n\cdot(n!)^2}{(-1)^n}\sinh^2\sqrt{-K}V. \tag{2.6}$$

Substituting (2.6) into above equality, we get (2.5).

Proof of Theorem 2.1 According to Definition 2.1, equality (2.5) may be written as

$$\sinh\sqrt{-K}V=\frac{1}{2^{\frac{n}{2}}\cdot(n!)}\Big(\prod_{i=1}^{n}\sinh\sqrt{-K}a_{0i}\Big)\cdot\sin P_0. \tag{2.7}$$

Now we only prove that

$$\frac{\prod\limits_{1\leqslant i<j\leqslant n}\sinh\sqrt{-K}a_{ij}}{\sin P_0}=\frac{\prod\limits_{0\leqslant i<j\leqslant n}\sinh\sqrt{-K}a_{ij}}{2^{\frac{n}{2}}\cdot n!\cdot\sinh\sqrt{-K}V}. \tag{2.8}$$

Applying (2.7), we get

$$\begin{aligned}\Big(\prod_{1\leqslant i<j\leqslant n}\sinh\sqrt{-K}a_{ij}\Big)\cdot\sinh\sqrt{-K}V &=\Big(\prod_{1\leqslant i<j\leqslant n}\sinh\sqrt{-K}a_{ij}\Big)\cdot\frac{1}{2^{\frac{n}{2}}\cdot(n!)}\prod_{i=1}^{n}\sinh\sqrt{-K}a_{0i}\cdot\sin P_0 \\ &=\frac{1}{2^{\frac{n}{2}}\cdot(n!)}\prod_{0\leqslant i<j\leqslant n}\sinh\sqrt{-K}a_{0i}\cdot\sin P_0.\end{aligned}$$

From above equality, we obtain (2.8).

Similarly, we can prove that other equalities in (2.3) also hold. The proof of Theorem 2.1 is completed.

3 The Law of Sine in Spherical Space

We consider the model of a spherical space in the Euclidean space (see [6]): the distance xy between two points x and y in the points set $S = \{x(x_1, x_2, \cdots, x_{n+1}) : x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = \frac{1}{K}, K > 0$ is constant number$\}$ in the $n+1$-dimensional Euclidean space E^{n+1} is the minimal non-negative real number satisfying

$$\cos\sqrt{K}xy = \frac{x_1y_1 + x_2y_2 + \cdots + x_{n+1}y_{n+1}}{\sqrt{x_1^2 + x_2^2 + \cdots + x_{n+1}^2}\sqrt{y_1^2 + y_2^2 + \cdots + y_{n+1}^2}}. \tag{3.1}$$

The metric space with this distance in the point set S is called n-dimensional spherical space with curvature $K > 0$, and denote by $S_n(K)$.

Let $\Omega_n(S)$ be an n-dimensional simplex in n-dimensional hyperbolic space $S^n(K)$, and $\{A_0, A_1, \cdots, A_n\}$ be its vertexes,a_{ij} $(0 \leqslant i, j \leqslant n)$ be its edge-length, V be its volume.

To give the law of sines in n-dimensional spherical space $S^n(K)$, we give the following definition.

Definition 3.1 Suppose that $\Omega_n(S) = \{A_0, A_1, \cdots, A_n\}$ be an n-dimensional simplex in n-dimensional spherical space $S^n(K)$, n edges A_0A_i $(i = 1, 2, \cdots, n)$ with initial point A_0 form an n-dimensional space angle A_0 of the simplex $\Omega_n(A)$. Denote by $\widehat{i,j}$ be the included angle between two edges A_0A_i and A_0A_j. The sine of the n-dimensional space angle A_0 of the simplex $\Omega_n(S)$ is defined as follows

$$\sin A_0 = (\det B_0)^{\frac{1}{2}}, \tag{3.2}$$

where

$$B_0 = \begin{bmatrix} 1 & & & \cos\widehat{i,j} \\ & 1 & & \\ & & \ddots & \\ \cos\widehat{i,j} & & & 1 \end{bmatrix} \quad (i, j = 1, 2, \cdots, n).$$

Similarly, we can define the sine of the n-dimensional space angle A_i $(i = 1, 2, \cdots, n)$ of the simplex $\Omega_n(S)$.

At first, we prove that this definition is sensible.

Actually, $\widehat{i,j}$ be the included angle between unit tangent vector $\vec{e_i}$ and $\vec{e_j}$ at point A_0 of two arcs $\widehat{A_0A_i}$ and $\widehat{A_0A_j}$. Because the unit tangent vectors $\vec{e_1}, \vec{e_2}, \cdots, \vec{e_n}$ are independence, the Gram matrix B_0 of the unit vectors $\vec{e_1}, \vec{e_2}, \cdots, \vec{e_n}$ is positive, and it is easy to know that $0 < \det B_0 \leqslant 1$. Therefore, this definition is sensible.

Remark Especially two-dimensional space angle of two-dimensional spherical simplex (that is spherical triangle) is just interior angle of spherical triangle. So the n-dimensional space angle of n-dimensional simplex in n-dimensional spherical space $H^n(K)$ is extension of interior angle of spherical triangle.

Definition 3.2 (see [11]) Suppose that $\Omega_n(S) = \{A_0, A_1, \cdots, A_n\}$ be an n-dimensional simplex in n-dimensional spherical space $S^n(K)$, its volume V is the minimal non-negative real number satisfying

$$\sin^2 \sqrt{K}V = \frac{1}{2^n \cdot n!^2}(\det\Lambda_n), \tag{3.3}$$

where $\Lambda_n(S) = (\cos\sqrt{K}a_{ij})_{i,j=0}^n$.

Theorem 3.1 Suppose that $\Omega_n(S) = \{A_0, A_1, \cdots, A_n\}$ be an n-dimensional simplex in n-dimensional spherical space $S^n(K)$, we have

$$\frac{\prod\limits_{1\leqslant i<j\leqslant n} \sin\sqrt{K}a_{ij}}{\sin A_0} = \frac{\prod\limits_{\substack{0\leqslant i<j\leqslant n\\ i,j\neq 1}} \sin\sqrt{K}a_{ij}}{\sin A_1} = \cdots = \frac{\prod\limits_{0\leqslant i<j\leqslant n-1} \sin\sqrt{K}a_{ij}}{\sin A_n} = \frac{\prod\limits_{0\leqslant i<j\leqslant n} \sin\sqrt{K}a_{ij}}{2^{\frac{n}{2}}\cdot n!\cdot \sin\sqrt{K}V}, \tag{3.4}$$

where definitions of $\sin A_i$ $(i = 0, 1, \cdots)$ are the same as Definition 3.1.

Remark When $n = 2$ in Theorem 3.1, it is the law of sine of a spherical triangle.

Lemma 3.2 (see [2]) (the law of cosine of a spherical triangle) For spherical triangle ABC in $S^2(1)$, then

$$\cos a = \cos b \cdot \cosh c + \sin b \cdot \sin c \cdot \cos A, \tag{3.5}$$

where a, b, c be edge-lengths of spherical triangle ABC and A be interior angle.

Lemma 3.3 Suppose that $\Omega_n(S) = \{A_0, A_1, \cdots, A_n\}$ be an n-dimensional simplex in n-dimensional spherical space $S^n(K)$, we have

$$\sin^2\sqrt{K}V = \frac{1}{2^n\cdot(n!)^2}\Big(\prod_{i=1}^n \sin^2\sqrt{K}a_{0i}\Big)\cdot \begin{vmatrix} 1 & & & \cos\widehat{i,j} \\ & 1 & & \\ & & \ddots & \\ \cos\widehat{i,j} & & & 1 \end{vmatrix}, \tag{3.6}$$

where $\widehat{i,j}$ $(i, j = 1, 2, \cdots, n)$ be the included angle between the edges P_0P_i and P_0P_j.

Proof Assume that the row or the column number of determinant $\det(\Lambda_n(S))$ begins from 0. Now, we transform the determinant $\det(\Lambda_n(S))$ as follows:

(1) for $i = 1, 2, \cdots, n$, plus $(-\cos\sqrt{K}a_{0i})$ times the 0-th row to the i-th row;

(2) expanding the determinant at the 0-th column;

(3) for $i = 1, 2, \cdots, n$, $\frac{1}{(\sin\sqrt{K}a_{0i})}$ times the i-th row and $\frac{1}{(\sin\sqrt{K}a_{0i})}$ times the i-th column.

$$\det(\Lambda_n(S)) = \begin{vmatrix} 1 & \cos\sqrt{K}a_{01} & \cdots & \cos\sqrt{K}a_{0n} \\ \cos\sqrt{K}a_{10} & 1 & \cdots & \cos\sqrt{K}a_{1n} \\ & \cdots & \cdots & \\ \cos\sqrt{K}a_{n0} & \cos\sqrt{K}a_{n1} & \cdots & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \cos\sqrt{K}a_{01} & \cdots & \cos\sqrt{K}a_{0n} \\ 0 & 1-\cos^2\sqrt{K}a_{01} & \cdots & \cos\sqrt{K}a_{1n} - \cos\sqrt{K}a_{0n}\cos\sqrt{K}a_{01} \\ & \cdots & \cdots & \cdots \\ 0 & \cos\sqrt{K}a_{n1} - \cos\sqrt{K}a_{0n}\cos\sqrt{K}a_{01} & \cdots & 1-\cos^2\sqrt{K}a_{0n} \end{vmatrix}$$

$$= \begin{vmatrix} 1-\cos^2\sqrt{K}a_{01} & \cdots & \cos\sqrt{K}a_{1n} - \cos\sqrt{K}a_{0n}\cos\sqrt{K}a_{01} \\ \cdots & \cdots & \cdots \\ \cos\sqrt{K}a_{n1} - \cos\sqrt{K}a_{0n}\cos\sqrt{K}a_{01} & \cdots & 1-\cos^2\sqrt{K}a_{0n} \end{vmatrix}$$

$$=\Big(\prod_{i=1}^{n}\sin^2\sqrt{K}a_{0i}\Big) \begin{vmatrix} \frac{1-\cos^2\sqrt{K}a_{01}}{\sin^2\sqrt{K}a_{01}} & \cdots & \frac{\cos\sqrt{K}a_{1n}-\cos\sqrt{K}a_{0n}\cos\sqrt{K}a_{01}}{\sin\sqrt{K}a_{01}\sin\sqrt{K}a_{0n}} \\ \cdots & \cdots & \cdots \\ \frac{\cos\sqrt{K}a_{n1}-\cos\sqrt{K}a_{0n}\cos\sqrt{K}a_{01}}{\sin\sqrt{K}a_{01}\sin\sqrt{K}a_{0n}} & \cdots & \frac{1-\cos^2\sqrt{K}a_{0n}}{\sin^2\sqrt{K}a_{0n}} \end{vmatrix}$$

$$=\Big(\prod_{i=1}^{n}\sin^2\sqrt{K}a_{0i}\Big) \begin{vmatrix} 1 & & & \cos\widehat{i,j} \\ & 1 & & \\ & & \ddots & \\ \cos\widehat{i,j} & & & 1 \end{vmatrix}.$$

By(3.3) we have

$$\det(\Lambda_n(S)) = 2^n \cdot (n!)^2 \sin^2\sqrt{K}V. \tag{3.7}$$

Substituting (3.7) into above equality, we get (3.6).

Proof of Theorem 3.1 According to Definition 3.1, equality (3.6) may be written as

$$\sin\sqrt{K}V = \frac{1}{2^{\frac{n}{2}}\cdot(n!)}\Big(\prod_{i=1}^{n}\sin\sqrt{K}a_{0i}\Big)\cdot\sin A_0. \tag{3.8}$$

Now we only prove that

$$\frac{\prod\limits_{1\leqslant i<j\leqslant n}\sin\sqrt{K}a_{ij}}{\sin A_0} = \frac{\prod\limits_{0\leqslant i<j\leqslant n}\sin\sqrt{K}a_{ij}}{2^{\frac{n}{2}}\cdot n!\cdot\sin\sqrt{K}V}. \tag{3.9}$$

Applying (3.8), we get

$$\begin{aligned}\Big(\prod_{1\leqslant i<j\leqslant n}\sin\sqrt{K}a_{ij}\Big)\cdot\sin\sqrt{K}V =& \Big(\prod_{1\leqslant i<j\leqslant n}\sin\sqrt{K}a_{ij}\Big)\cdot\frac{1}{2^{\frac{n}{2}}\cdot n!}\prod_{i=1}^{n}\sin\sqrt{K}a_{0i}\cdot\sin A_0 \\ =& \frac{1}{2^{\frac{n}{2}}\cdot(n!)}\Big(\prod_{0\leqslant i<j\leqslant n}\sin\sqrt{K}a_{0i}\Big)\cdot\sin A_0.\end{aligned}$$

From above equality, we obtain (3.9).

Similarly, we can prove that other equalities in (3.4) also hold. The proof of Theorem 3.1 is completed.

4 Some Geometric Inequalities

On basis of Section 2 and Section 3, we are easy to establish Veljan-Korchmaros type inequalities and Hadamard type inequalities in the n-dimensional hyperbolic space $H^n(K)$ and the n-dimensional spherical space $S^n(K)$. In addition, some new geometric inequality about "metric addition" [11, 14] involving Volume and n-dimensional angle of simplex in $H^n(K)$ and $S^n(K)$ is established.

Theorem 4.1 Suppose that $\Sigma_n(H)=\{P_0,P_1,\cdots,P_n\}$ be an n-dimensional simplex in n-dimensional hyperbolic space $H^n(K)$, we have

$$\prod_{0\leq i<j\leq n}\sinh\sqrt{-K}a_{ij}\geqslant 2^{\frac{n(n+1)}{2}}(n!)^{n+1}(\sinh\sqrt{-K}V)^{n+1}. \tag{4.1}$$

Proof Because $\sin P_i\leqslant 1$ in (2.3), we have

$$\sinh^2\sqrt{-K}V\leqslant\frac{1}{2^n\cdot(n!)^2}\prod_{j=1}^{n}\sinh\sqrt{-K}a_{ij},\quad j=0,1,\cdots,n. \tag{4.2}$$

Multiplying by above those inequalities for $j=0,1,\cdots,n$, we get (4.1).

Theorem 4.2 Suppose that $\Omega_n(S)=\{A_0,A_1,\cdots,A_n\}$ be an n-dimensional simplex in n-dimensional spherical space $S^n(K)$, we have

$$\prod_{0\leqslant i<j\leqslant n}\sin\sqrt{K}a_{ij}\geqslant 2^{\frac{n(n+1)}{2}}(n!)^{n+1}(\sin\sqrt{K}V)^{n+1}. \tag{4.3}$$

Proof Because $\sin A_i\leqslant 1$ in (3.3), we have

$$\sin^2\sqrt{K}V\leqslant\frac{1}{2^n\cdot(n!)^2}\prod_{j=1}^{n}\sin\sqrt{K}a_{ij},\quad j=0,1,\cdots,n. \tag{4.4}$$

Multiplying by above those inequalities for $j=0,1,\cdots,n$, we get (4.3).

Since $\sin P_0\leqslant 1$ in (2.7) and $\sin A_0\leqslant 1$ in (3.7), thus we obtain Hadamard type inequalities in n-dimensional hyperbolic space $H^n(K)$ and n-dimensional spherical space $S^n(K)$ as follows:

Theorem 4.3 Suppose that $\Sigma_n(H)=\{P_0,P_1,\cdots,P_n\}$ be an n-dimensional simplex in n-dimensional hyperbolic space $H^n(K)$, we have

$$\sinh\sqrt{-K}V\leqslant\frac{1}{2^{\frac{n}{2}}\cdot(n!)}\prod_{i=1}^{n}\sinh\sqrt{-K}a_{0i}. \tag{4.5}$$

Theorem 4.4 Suppose that $\Omega_n(S)=\{A_0,A_1,\cdots,A_n\}$ be an n-dimensional simplex in n-dimensional spherical space $S^n(K)$, we have

$$\sin\sqrt{K}V\leqslant\frac{1}{2^{\frac{n}{2}}\cdot(n!)}\prod_{i=1}^{n}\sin\sqrt{K}a_{0i}. \tag{4.6}$$

Theorem 4.5 Let $\Sigma''_n(H)$ be n-dimensional metric addition simplex which is formed two n-dimensional simplexes $\Sigma_n(H)$ and $\Sigma'_n(H)$ by "metric addition" operation in n-dimensional hyperbolic space $H^n(K)$, we have

$$\Big(\frac{\sinh\sqrt{-K}V''}{\sin P''_i}\Big)^{\frac{1}{n}} \geqslant \Big(\frac{\sinh\sqrt{-K}V}{\sin P_i}\Big)^{\frac{1}{n}} + \Big(\frac{\sinh\sqrt{-K}V'}{\sin P'_i}\Big)^{\frac{1}{n}} \quad (i=0,1,\cdots,n). \tag{4.7}$$

Equality obtain if and only if the simplex $\Sigma_n(H)$ and $\Sigma'_n(H)$ is regular.

Theorem 4.6 Let $\Omega''_n(S)$ be n-dimensional metric addition simplex which is formed two n-dimensional simplexes $\Omega_n(S)$ and $\Omega'_n(S)$ by "metric addition" operation in n-dimensional spherical space $S^n(K)$, we have

$$\Big(\frac{\sin\sqrt{K}V''}{\sin A''_i}\Big)^{\frac{1}{n}} \geqslant \Big(\frac{\sin\sqrt{K}V}{\sin A_i}\Big)^{\frac{1}{n}} + \Big(\frac{\sin\sqrt{K}V'}{\sin A'_i}\Big)^{\frac{1}{n}} \quad (i=0,1,\cdots,n). \tag{4.8}$$

Equality obtain if and only if the simplex $\Omega_n(S)$ and $\Omega'_n(S)$ is regular.

Lemma 4.7 (see [15]) Let $a_k, b_k \geq 0$, then

$$\prod_{k=1}^{n}(a_k+b_k)^{\frac{1}{n}} \geq \Big(\prod_{k=1}^{n} a_k\Big)^{\frac{1}{n}} + \Big(\prod_{k=1}^{n} b_k\Big)^{\frac{1}{n}}. \tag{4.9}$$

The Proof of Theorem 4.5 According to the definition of "metric addition" [11] in $H^n(K)$, we have $\sinh\sqrt{-K}a''_{0i} = \sinh\sqrt{-K}a_{0i} + \sinh\sqrt{-K}a_{0i}$ (for $i=1,2,\cdots,n$). Thus

$$\prod_{i=1}^{n}\sinh\sqrt{-K}a''_{0i} = \prod_{i=1}^{n}(\sinh\sqrt{-K}a_{0i} + \sinh^2\sqrt{-K}a'_{0i})$$

$\frac{1}{n}$-th power on the both sides, we get

$$\Big(\prod_{i=1}^{n}\sinh\sqrt{-K}a''_{0i}\Big)^{\frac{1}{n}} = \Big[\prod_{i=1}^{n}(\sinh\sqrt{-K}a_{0i} + \sinh\sqrt{-K}a'_{0i})\Big]^{\frac{1}{n}}. \tag{4.10}$$

By (4.9) and (4.10), we have

$$\begin{aligned}\Big(\prod_{i=1}^{n}\sinh\sqrt{-K}a''_{0i}\Big)^{\frac{1}{n}} &= \Big[\prod_{i=1}^{n}(\sinh\sqrt{-K}a_{0i} + \sinh\sqrt{-K}a'_{0i})\Big]^{\frac{1}{n}} \\ &\geqslant \Big(\prod_{i=1}^{n}\sinh\sqrt{-K}a_{0i}\Big)^{\frac{1}{n}} + \Big(\prod_{i=1}^{n}\sinh\sqrt{-K}a'_{0i}\Big)^{\frac{1}{n}}. \end{aligned} \tag{4.11}$$

By (2.3), we obtain

$$\Big(\frac{\sinh\sqrt{-K}V''}{\sin P''_0}\Big)^{\frac{2}{n}} \geqslant \Big(\frac{\sinh\sqrt{-K}V}{\sin P_0}\Big)^{\frac{2}{n}} + \Big(\frac{\sinh\sqrt{-K}V'}{\sin P'_0}\Big)^{\frac{2}{n}}.$$

Similarly, inequality (4.7) is easy proved for $i=1,2,\cdots,n$.

The proof of Theorem 4.6 is the same as the proof of Theorem 4.5.

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n维双曲空间和n维球面空间中的正弦定理及应用

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摘要: 本文研究了n维双曲空间和n维球面空间中单形的正弦定理和相关几何不等式. 应用距离几何的理论和方法, 给出了n维双曲空间和n维球面空间中一种新形式的正弦定理, 利用建立的正弦定理获得了Hadamard 型和Veljan-Korchmaros型不等式. 另外, 建立了涉及两个n维双曲单形和n维球面单形的“度量加”的一些几何不等式.

关键词: 双曲空间; 球面空间; 正弦定理; 度量加; 几何不等式

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