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OPTIMAL EIGHTH-ORDER STEFFENSEN TYPE METHODS FOR NONLINEAR EQUATIONS

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Abstract: In this paper, we present a family of three-step eighth-order Steffensen type methods for solving nonlinear equations by using weight function methods. Numerical experiments show that these methods require less time to converge than the other optimal methods.

Keywords: Steffensen's method; derivative free; eighth-order convergence

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1 Introduction

In this paper, we consider iterative methods to find a simple root of a nonlinear equation $f(x) = 0$, where $f : I \subset R \rightarrow R$ for an open interval I is a scalar function. The classical Newton's method [1] with second-order convergence is written as

$$x_{n+1} = x_n - f(x_n)f'(x_n)^{-1}. \tag{1.1}$$

However, when the first order derivative of the function $f(x)$ is unavailable or is expensive to compute, the Newton's method is still restricted in practical applications. In order to avoid computing the first order derivative, Steffensen [2] proposed the following second-order method

$$x_{n+1} = x_n - f(x_n)f[x_n, z_n]^{-1}, \tag{1.2}$$

where $z_n = x_n + f(x_n)$, and $f[\cdot\ ,\ \cdot]$ is the first order divided difference. To improve the local order of convergence, many high-order methods were proposed in open literatures, see [3–12] and references therein. Ren et al. [3] proposed the following fourth-order methods

$$\begin{cases} y_n = x_n - f(x_n)f[x_n, z_n]^{-1}, \\ x_{n+1} = y_n - f(y_n)[f[x_n, y_n] + f[y_n, z_n] - f[x_n, z_n] + \beta(y_n - x_n)(y_n - z_n)]^{-1}, \end{cases} \tag{1.3}$$

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where $z_n = x_n + f(x_n)$ and $\beta \in R$ is a constant. Zheng et al. [4] presented an eighth-order method by using the direct Newtonian interpolation, which is given by

$$\begin{cases} y_n = x_n - f(x_n) f[x_n, z_n]^{-1}, \\ u_n = y_n - f(y_n)\{f[x_n, y_n] + f[z_n, x_n, y_n](y_n - x_n)\}^{-1}, \\ x_{n+1} = u_n - f(u_n)\{f[u_n, y_n] + f[u_n, x_n, y_n](u_n - y_n) \\ \qquad + f[u_n, z_n, x_n, y_n](u_n - y_n)(u_n - x_n)\}^{-1}, \end{cases} \tag{1.4}$$

where $z_n = x_n + \gamma f(x_n)$ and $\gamma \in R$ is a constant. Furthermore, Soleymani et al. [5] also presented the following eight-order method

$$\begin{cases} y_n = x_n - f(x_n) f[x_n, z_n]^{-1}, \\ u_n = y_n - f(y_n) f[x_n, y_n]^{-1} \left\{1 + t_n + t_n^2 - t_n^3/2\right\}, \\ x_{n+1} = u_n - f(u_n) f[u_n, y_n]^{-1} \left\{1 - (f[x_n, z_n] - 1)^{-1} t_n^2 + (2 - f[x_n, z_n])\lambda_n\right\}, \end{cases} \tag{1.5}$$

where $z_n = x_n - f(x_n), t_n = f(y_n)/f(z_n)$ and $\lambda_n = f(u_n)/f(z_n)$. Other Steffensen type methods and their applications were discussed in [6–12]. All these methods are derivative-free in per full iteration.

The purpose of this paper is to develop a new family of eighth-order derivative-free methods and give the convergence analysis. This paper is organized as follows. In Section 2, we present a family of three-step eighth-order iterative methods for solving nonlinear equations. The new methods are free from any derivatives and require four evaluations of the function $f(x)$, therefore the new methods have the efficiency index of $\sqrt[4]{8} \approx 1.682$. The new methods agree with the conjecture of Kung and Traub [13] for the case $n = 4$. We prove that the order of convergence of the new methods is eight for nonlinear equations. In Section 3, we give some specific iterative methods which can be used in practical computations. Numerical examples are given in Section 4 to illustrate convergence behavior of our methods for simple roots. Section 5 is a short conclusion.

2 The Methods and Analysis of Convergence

Now, we consider the iteration scheme of the form,

$$\begin{cases} y_n = x_n - f(x_n) f[x_n, z_n]^{-1}, \\ u_n = y_n - K(s_n, t_n) f(y_n) f[x_n, z_n]^{-1}, \\ x_{n+1} = u_n - H(\lambda_n) f(u_n) f'(u_n)^{-1}, \end{cases} \tag{2.1}$$

where $z_n = x_n + \gamma f(x_n)$, $s_n = f(y_n)/f(x_n)$, $t_n = f(y_n)/f(z_n)$, $\lambda_n = f(u_n)/f(z_n)$ and $\gamma \in R$ is a constant. $H(\lambda)$ and $K(s, t)$ are some functions of one and two variables, respectively. It is quite obvious that formula (2.1) requires five functional evaluations per iteration. To derive a scheme with a higher efficiency index and reduce the number of functional evaluations, we approximate $f'(u_n)$ by considering the approximation of $f(x)$ by a rational nonlinear function of the form

$$\phi(x) = \{a_1 + a_2(x - x_n)\}\{1 + a_3(x - x_n)\}^{-1}, \tag{2.2}$$

where the parameters a_1, a_2 and a_3 are determined by the condition that f and ϕ coincide at x_n, y_n and u_n. That means $\phi(x)$ satisfies the conditions

$$\phi(x_n) = f(x_n),\ \phi(y_n) = f(y_n),\ \phi(u_n) = f(u_n). \tag{2.3}$$

From (2.2) and (2.3), we can obtain

$$a_1 = f(x_n), \tag{2.4}$$
$$a_2 = \{f[x_n, u_n]f(y_n) - f[x_n, y_n]f(u_n)\}\{f(y_n) - f(u_n)\}^{-1}, \tag{2.5}$$
$$a_3 = f[x_n, u_n] - f[x_n, y_n]\{f(y_n) - f(u_n)\}^{-1}. \tag{2.6}$$

Differentiation of (2.2) gives

$$\phi'(x) = (a_2 - a_1a_3)(1 + a_3(x - x_n))^{-2}. \tag{2.7}$$

We can now approximate the derivative $f'(x)$ with the derivative $\phi'(x)$ and obtain

$$f'(u_n) \approx \phi'(u_n). \tag{2.8}$$

From (2.4) and (2.5)–(2.8), it follows that

$$f'(u_n) \approx f(x_n)f[x_n, u_n]f[u_n, y_n](f(x_n) - f(y_n))^{-1}f[x_n, z_n]^{-1}. \tag{2.9}$$

Substituting (2.9) into (2.1), we obtain a new family of eighth-order methods:

$$\begin{cases} y_n = x_n - f(x_n)f[x_n, z_n]^{-1}, \\ u_n = y_n - K(s_n, t_n)f(y_n)f[x_n, z_n]^{-1}, \\ x_{n+1} = u_n - H(\lambda_n)f[x_n, z_n]f(u_n)(1 - s_n)\{f[u_n, x_n]f[u_n, y_n]\}^{-1}. \end{cases} \tag{2.10}$$

The functions $H(\lambda)$ and $K(s, t)$ should be determined so that the iterative method (2.10) is of the order eight. To do that, we will use the Taylor's series about (0) for $H(\lambda)$ and $(0, 0)$ for $K(s, t)$ thus,

$$H(\lambda) = H(0) + H'(0)\lambda + \cdots, \tag{2.11}$$
$$K(s, t) = K(0, 0) + K_s s + K_t t + [K_{ss}s^2 + 2K_{st}st + K_{tt}t^2]/2 + \cdots. \tag{2.12}$$

Here the subscribes denote respective partial derivatives; for example, $K_{st} = \left.\frac{\partial^2 K(s,t)}{\partial s \partial t}\right|_{(s,t)=(0,0)}$, etc. Petković et al. [10] proved that the first two steps of (2.10) are fourth order methods with $K(0, 0) = 1,\ K_s = 1$ and $K_t = 1$. We could find another coefficients $H(0),\ H'(0),\ K_{ss}, K_{st},\ K_{tt}$ by Theorem 2.1.

Theorem 2.1 Let $a \in I$ be a simple zero of a sufficiently differentiable function $f: I \in R \to R$ for an open interval I. If x_0 is sufficiently close to a, then the sequence $\{x_n\}$ generated by any method of the family (2.10) converges to a. If $H(0)=1,\ H'(0)=1,\ |H''(0)|<\infty, K(0,0)=1, K_s=1,\ K_t=1,\ K_{ss}=2,\ K_{tt}=2, K_{st}=2$, then the family of methods defined by (2.10) is of eighth-order.

Proof Let $e_n = x_n - a,\ c_n = (1/n!)f^{(n)}(a)/f'(a),\ n=2,3,\cdots$. Using the Taylor expansion and taking into account $f(a)=0$, we have

$$f(x_n) = f'(a)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + O(e_n^7)]. \tag{2.13}$$

From (2.13), noting that $z_n = x_n + \gamma f(x_n)$, we can obtain

$$\begin{aligned} f(z_n) &= f'(a)[(1+\gamma f'(a))e_n + c_2(\gamma f'(a) + (1+\gamma f'(a))^2)e_n^2 \\ &\quad +(c_3(\gamma f'(a) + (1+\gamma f'(a))^3) + 2\gamma f'(a)(1+\gamma f'(a))c_2^2)e_n^3 + (c_2^3\gamma^2 f'(a)^2 \\ &\quad +c_2c_3\gamma f'(a)(1+\gamma f'(a))(5+3\gamma f'(a)) \\ &\quad +c_4(\gamma f'(a) + (1+\gamma f'(a))^4))e_n^4 + O(e_n^5), \end{aligned} \tag{2.14}$$

$$\begin{aligned} f[x_n, z_n] &= f'(a) + (2+\gamma f'(a))c_2 f'(a)e_n + (3c_3 + (c_2^2+3c_3)\gamma f'(a) \\ &\quad +c_3\gamma^2 f'(a)^2)f'(a)e_n^2 + (c_4(4+6\gamma f'(a) + 4\gamma^2 f'(a)^2 + \gamma^3 f'(a)^3) \\ &\quad +c_2c_3(4\gamma f'(a) + 2\gamma^2 f'(a)^2))f'(a)e_n^3 + O(e_n^4), \end{aligned} \tag{2.15}$$

then,

$$y_n - a = Ae_n^2 + Be_n^3 + Ce_n^4 + O(e_n^5), \tag{2.16}$$

where

$$A = c_2(1+\gamma f'(a)), \tag{2.17}$$

$$B = c_3(2+3\gamma f'(a) + \gamma^2 f'(a)^2) - c_2^2(2+2\gamma f'(a) + \gamma^2 f'(a)^2), \tag{2.18}$$

$$\begin{aligned} C &= c_4(3+6\gamma f'(a) + 4\gamma^2 f'(a)^2 + \gamma^3 f'(a)^3) - c_2c_3(7+10\gamma f'(a) \\ &\quad +7\gamma^2 f'(a)^2 + 2\gamma^3 f'(a)^3) + c_2^3(4+5\gamma f'(a) + 3\gamma^2 f'(a)^2 + \gamma^3 f'(a)^3). \end{aligned} \tag{2.19}$$

With (2.13), (2.14) and (2.16), we have

$$\begin{aligned} f(y_n) &= f'(a)[y_n - a + c_2(y_n-a)^2 + O(e^5)] \\ &= f'(a)[Ae_n^2 + Be_n^3 + (C+A^2)e_n^4 + O(e_n^5)], \end{aligned} \tag{2.20}$$

$$\begin{aligned} s_n &= f'(a) + f'(a)c_2e_n + f'(a)(c_2^2 + c_3 + \gamma f'(a)c_2^2)e_n^2 + f'(a)(-2c_2^3 + 3c_2c_3 + c_4 \\ &\quad -2\gamma f'(a)c_2^3 + 4\gamma f'(a)c_2c_3 + (c_2c_3 - c_2^3)\gamma^2 f'(a)^2)e_n^3 + f'(a)(4c_2^4 - 8c_2^2c_3 + 2c_3^2 \\ &\quad +4c_2^4 + c_5 + \gamma f'(a)(5c_2^4 - 10c_2^2c_3 + 3c_3^2 + 7c_2c_4) + \gamma^2 f'(a)^2(3c_2^4 - 7c_2^2c_3 \\ &\quad +c_3^2 + 4c_2c_4) + \gamma^3 f'(a)^3(c_2^4 - 2c_2^2c_3 + c_2c_4))e_n^4 + O(e_n^5), \end{aligned} \tag{2.21}$$

$$\begin{aligned} t_n &= (1+\gamma f'(a))c_2e_n + (-3c_2^2 + 2c_3 + \gamma f'(a)(3c_3 - 3c_2^2) \\ &\quad +\gamma^2 f'(a)^2(c_3 - c_2^2))e_n^2 + (8c_2^3 - 10c_2c_3 + 3c_4 + \gamma f'(a)(10c_2^3 - 14c_2c_3 + 6c_4) \\ &\quad +(5c_2^3 - 8c_2c_3 + 4c_4)\gamma^2 f'(a)^2 + \gamma^3 f'(a)^3(c_2^3 - 2c_2c_3 + c_4))e_n^3 \end{aligned}$$

$$\begin{aligned}
&+(-20c_2^4+37c_2^2c_3-8c_3^2-14c_2c_4+4c_5+\gamma f'(a)\\
&(60c_2^2c_3-30c_2^4-15c_3^2-25c_2c_4+10c_5)+\gamma^2 f'(a)^2(44c_2^2c_3-20c_2^4-13c_3^2-20c_2c_4+10c_5)\\
&+\gamma^3 f'(a)^3(17c_2^2c_3-7c_2^4-6c_3^2-9c_2c_4+5c_5)\\
&+\gamma^4 f'(a)^4(3c_2^2c_3-c_2^4-c_3^2-2c_2c_4+c_5))e_n^4+O(e_n^5). \qquad (2.22)
\end{aligned}$$

Using the Taylor expansion (2.12) with $K(0,0)=K_s=K_t=1$, we get

$$K(s_n,t_n)=1+s_n+t_n+[K_{ss}s_n^2+2K_{st}s_nt_n+K_{tt}t_n^2]/2+O(e_n^5). \qquad (2.23)$$

Together with (2.15)–(2.23), we have

$$u_n-a=De_n^4+Ee_n^5+O(e_n^6), \qquad (2.24)$$

where

$$\begin{aligned}
D &= -1/2c_2(1+\gamma f'(a))(2c_3(1+\gamma f'(a))+c_2^2(-10+K_{tt}+2K_{st}+K_{ss}\\
&\quad +\gamma f'(a)(-10+2K_{st}+2K_{ss})+\gamma^2 f'(a)^2(-2+K_{ss})), \qquad (2.25)\\
E &= 1/2(-2c_2c_4(1+\gamma f'(a))^2(2+\gamma f'(a))-2c_3^2(2+\gamma f'(a))(1+\gamma f'(a))^2\\
&\quad -c_2^2c_3(1+\gamma f'(a))(-64+6K_{tt}+\gamma f'(a)(15K_{ss}-92)+\gamma^2 f'(a)^2(-44+12K_{ss})\\
&\quad +\gamma^3 f'(a)^3(-6+3K_{ss})+3K_{tt}(2+\gamma f'(a))+6K_{st}(2+3\gamma f'(a)+\gamma^2 f'(a)^2))\\
&\quad +c_2^4(-72+10K_{ss}+\gamma f'(a)(-160+31K_{ss})+\gamma^2 f'(a)^2(-132+36K_{ss})\\
&\quad +\gamma^3 f'(a)^3(-48+19K_{ss})+\gamma^4 f'(a)^4(-6+4K_{ss})+K_{tt}(10+15\gamma f'(a)+6\gamma^2 f'(a)^2)\\
&\quad +2K_{st}(10+23\gamma f'(a)+18\gamma^2 f'(a)^2+5\gamma^3 f'(a)^3)))e_n^5+O(e_n^6). \qquad (2.26)
\end{aligned}$$

Using (2.13), (2.14), (2.16), (2.20) and (2.24), we have

$$\begin{aligned}
f(u_n) &= f'(a)(De_n^4+Ee_n^5+O(e_n^6)), \qquad (2.27)\\
f[x_n,u_n] &= f'(a)[1+c_2e_n+c_3e_n^2+c_4e_n^3+(c_5+c_2D)e_n^4+O(e_n^5)], \qquad (2.28)\\
f[u_n,y_n] &= f'(a)[1+c_2Ae_n^2+c_2Be_n^3+((C+D)c_2+c_3A^2)e_n^4+O(e_n^5)], \qquad (2.29)\\
f[u_n,a] &= f'(a)+f''(a)(u_n-a)/2+O(e_n^6)\\
&= f'(a)[1+c_2De_n^4+c_2Ee_n^5+O(e_n^6)], \qquad (2.30)\\
\lambda_n &= D(1+\gamma f'(a))^{-1}e_n^3+[E(1+\gamma f'(a))\\
&\quad -c_2D(1+3\gamma f'(a)+\gamma^2 f'(a)^2)](1+\gamma f'(a))^{-2}e_n^4+O(e_n^5). \qquad (2.31)
\end{aligned}$$

With (2.15), (2.21), and (2.28)–(2.30), we have

$$\begin{aligned}
\frac{(1-s_n)f[u_n,a]f[x_n,z_n]}{f[u_n,x_n]f[u_n,y_n]} &= 1+(c_3-c_2^2)Ae_n^3+(c_2^3(2+3\gamma f'(a)+\gamma^2 f'(a)^2)\\
&\quad +c_2(c_4-D+c_4\gamma f'(a))+c_2^4(3+3\gamma f'(a)+\gamma^2 f'(a)^2)\\
&\quad -c_2^2c_3(6+7\gamma f'(a)+2\gamma^2 f'(a)^2))e_n^4+O(e_n^5). \qquad (2.32)
\end{aligned}$$

Hence, together with (2.11), (2.24) and (2.32), we obtain

$$\begin{aligned} e_{n+1} \;=\;& (u_n - a)\big(1 - H(0) + ((c_3 - c_2^2)AH(0) - DH'(0)\{1+\gamma f'(a)\}^{-1})e_n^3 \\ &+(-(c_2(c_4 - D + c_4\gamma f'(a)) + c_3^2(2+3\gamma f'(a)+\gamma^2 f'(a)^2) + c_2^4(3+3\gamma f'(a) \\ &+\gamma^2 f'(a)^2) - c_2^2 c_3(6+7\gamma f'(a)+2\gamma^2 f'(a)^2))H(0) + H'(0)[c_2 D(1+3\gamma f'(a) \\ &+\gamma^2 f'(a)^2) - (1+\gamma f'(a))E](1+\gamma f'(a))^{-2})e_n^4 + O(e_n^5)\big). \end{aligned} \tag{2.33}$$

Now, using (2.33) with $H(0)=1,\ H'(0)=1, K(0,0)=1, K_s=1, K_t=1, K_{ss}=2, K_{tt}=2$ and $K_{st}=2$, we obtain the error equation

$$\begin{aligned} e_{n+1} \;=\;& (1+\gamma f'(a))^2 c_2^2 (c_3 - c_2^2)[(4c_2c_3 - c_4)(1+\gamma f'(a))^2 \\ &+c_2^3(5+6\gamma f'(a)+3\gamma^2 f'(a)^2+\gamma^3 f'(a)^3)]e_n^8 + O(e_n^9). \end{aligned} \tag{2.34}$$

This means that the convergence order of any method of family (2.10) is eighth-order, when $K(0,0)=K_s=K_t=1,\ K_{ss}=K_{tt}=K_{st}=2,\ H(0)=H'(0)=1,$ and $|H''(0)|<\infty$. The proof is completed.

3 The Concrete Iterative Methods

The functions $H(\lambda)$ and $K(s,t)$ can take many forms satisfying the conditions of Theorem 2.1. In order to reduce the computational cost, we should choose $H(\lambda)$ and $K(s,t)$ as simple as possible. In what follows, we give some iterative forms:

$$\begin{aligned} &H_1(\lambda) = 1+\lambda+\eta\lambda^2,\ H_2(\lambda) = \{1+\lambda(\eta+1)\}(1+\eta\lambda)^{-1}, \eta\in R, \\ &K_1(s,t) = (1-s-t)^{-1},\ K_2(s,t) = 1+s+t+(s+t)^2. \end{aligned}$$

Method 1 Taking $H_1(\lambda)(\eta=0)$ and $K_1(s,t)$ into the iterative formula (2.10), we get a family of eighth-order methods

$$\begin{cases} y_n = x_n - f(x_n)f[x_n,z_n]^{-1}, \\ u_n = y_n - (1-s_n-t_n)^{-1} f(y_n)f[z_n,x_n]^{-1}, \\ x_{n+1} = u_n - (1+\lambda_n)(1-s_n) f[x_n,z_n]f(u_n)\{f[u_n,x_n]f[u_n,y_n]\}^{-1}, \end{cases} \tag{3.1}$$

where $z_n = x_n+\gamma f(x_n), s_n = \frac{f(y_n)}{f(x_n)}, t_n = \frac{f(y_n)}{f(z_n)}, \lambda_n = \frac{f(u_n)}{f(z_n)}$, and $\gamma\in R$ is a constants.

Method 2 Taking $H_2(\lambda)(\eta=-1)$ and $K_2(s,t)$ into the iterative formula (2.10), we get another family of eighth-order methods

$$\begin{cases} y_n = x_n - f(x_n)f[x_n,z_n]^{-1}, \\ u_n = y_n - \left(1+s_n+t_n+(s_n+t_n)^2\right) f(y_n)f[z_n,x_n]^{-1}, \\ x_{n+1} = u_n - f(z_n)f(u_n)(1-s_n) f[x_n,z_n]\{f[u_n,x_n]f[u_n,y_n](f(z_n)-f(u_n))\}^{-1}, \end{cases} \tag{3.2}$$

where $z_n = x_n+\gamma f(x_n), s_n = \frac{f(y_n)}{f(x_n)},\ t_n = \frac{f(y_n)}{f(z_n)}, \lambda_n = \frac{f(u_n)}{f(z_n)}$, and $\gamma\in R$ is a constants.

In terms of computational cost, the developed methods require evaluations of four functions per iteration. Consider the definition of efficiency index [14] as $p^{1/w}$, where p is the order of the method and w is the number of function evaluations per iteration required by the method. The new methods have the efficiency index of $\sqrt[4]{8} \approx 1.682$, which is higher than $\sqrt{2} \approx 1.414$ of Steffensen's method (1.2), $\sqrt[3]{4} \approx 1.587$ of Ren's method (1.3).

4 Numerical Results

In this section, all experiments have been carried out on a personal computer equipped with an Intel(R) Celeron(R) 430 CPU, 1.79 GHz and WinXp 32-bit operating system. Using the symbolic computation in the programming package Matlab 7.0 Now, Method 1 (M81, (3.1)) and Method 2 (M82, (3.2)) are employed to solve some nonlinear equations and compared with Steffensen's method (S2, (1.2)), Ren's method (R4, (1.3))($\beta = 1$), Zheng's method (Z8, (1.4)) with $\gamma = 1$ and Soleymani's method (S8,(1.5)). Table 1 shows the absolute values $|x_k - x_{k-1}|(k = 1, 2, \cdots 7)$ and the approximation x_n to a, where a is computed with 2400 significant digits and x_n is calculated by using the same total number of function evaluation (TNFE) for all methods. The absolute values of function ($|f(x_n)|$) and the computational order of convergence ρ are also shown in Table 1. Here, the TNFE for all methods is 12. The computational order of convergence ρ [15] is defined by

$$\rho \approx \ln(|(x_{n+1} - x_n)/(x_n - x_{n-1})|)\{\ln(|(x_n - x_{n-1})/(x_{n-1} - x_{n-2})|)\}^{-1}.$$

Following functions are used:

$$f_1(x) = \cos(x) - xe^x + x^2,\ a \approx 0.639154,\ x_0{=}0.5,$$
$$f_2(x) = \sqrt{x} - 1/x - 3,\ a \approx 9.633596,\ x_0 = 8,$$
$$f_3(x) = xe^{x^3} - 4x - 2,\ a \approx -0.622256,\ x_0{=}-0.5,$$
$$f_4(x) = \ln^{(-x^2+x+2)} -x + 1,\ a \approx 1.384123,\ x_0 = 1.$$

On the other hand, in Table 2 the mean elapsed time, after 100 performances of the program, appears. The stopping criterion used is $|x_{k+1} - x_k| + |f(x_k)| < 10^{-300}$.

5 Conclusions

By theoretical analysis and numerical experiments, we confirm that the new Steffensen type methods only use four evaluations of the function per iteration to achieve eighth-order convergence for solving a simple root of nonlinear functions. The new methods are free from any derivatives. The efficiency index of the new methods is $\sqrt[4]{8} \approx 1.682$. Table 1 show that the results of the new methods are similar to that of the other eighth-order optimal methods. Table 2 show that the new methods require less time to converge than the other optimal methods. Thus, our methods in this contribution can be considered as improvements of Steffensen's method.

Table 1 Comparison of various iterative methods (TNFE=12)

$f_i(x)$	Method	$\lvert x_1 - x_0\rvert$	$\lvert x_2 - x_1\rvert$	$\lvert x_3 - x_2\rvert$	$\lvert x_4 - x_3\rvert$	$\lvert x_5 - x_4\rvert$	$\lvert x_6 - x_5\rvert$	$\lvert x_7 - x_6\rvert$	ρ	$\lvert f(x_n)\rvert$
f_1	S2	.12	.17e-1	.34e-3	.13e-6	.19e-13	.40e-27	.18e-54	2.0	.43e-54
	R4	.14	.90e-4	.84e-17	.12e-66	.79e-268			4.0	.19e-267
	Z8	.14	.83e-8	.30e-65	.77e-525				8.0	.19e-524
	S8	.14	.17e-4	.25e-37	.44e-300				8.0	.11e-299
	M81	.14	.67e-8	.41e-66	.81e-532				8.0	**.20e-531**
	M82	.14	.71e-8	.70e-66	.68e-530				8.0	.16e-529
f_2	S2	1.5	.11	.44e-3	.71e-8	.18e-17	.12e-36	.50e-75	2.0	.85e-76
	R4	1.5	.10	.27e-4	.12e-18	.56e-76			4.0	.97e-77
	Z8	1.6	.27e-7	.67e-70	.90e-571				8.0	.15e-571
	S8	1.6	.23e-9	.14e-88	.34e-722				8.0	.58e-723
	M81	1.6	.88e-10	.61e-92	.33e-749				8.0	**.56e-750**
	M82	1.6	.21e-7	.16e-70	.20e-575				8.0	.35e-576
f_3	S2	.11	.14e-1	.21e-3	.49e-7	.27e-14	.78e-29	.67e-58	2.0	.25e-57
	R4	.12	.80e-4	.30e-16	.57e-66	.78e-265			4.0	.29e-264
	Z8	.12	.12e-7	.35e-64	.23e-516				8.0	.89e-516
	S8	.12	.98e-7	.58e-55	.85e-441				8.0	.32e-440
	M81	.12	.10e-7	.11e-64	.22e-520				8.0	.82e-520
	M82	.12	.79e-8	.14e-65	.16e-527				8.0	**.62e-527**
f_4	S2	.30	.75e-1	.44e-2	.15e-4	.18e-9	.24e-19	.46e-39	2.0	.10e-38
	R4	.38	.61e-2	.86e-9	.35e-36	.94e-146			4.0	.21e-145
	Z8	.38	.13e-4	.79e-40	.20e-321				8.0	.43e-321
	S8	.38	.35e-3	.36e-27	.46e-219				8.0	.10e-218
	M81	.38	.87e-6	.32e-50	.13e-405				8.0	**.28e-405**
	M82	.38	.88e-6	.37e-50	.33e-405				8.0	.74e-405

Table 2 Mean e-time in 100 performances of the program.

f	x_0	S2	R4	Z8	S8	M81
f_1	0.5	3.4844	3.5725	3.2906	7.2666	2.9950
f_2	8	1.1819	1.3313	1.4244	1.4595	1.2161
f_3	-0.5	2.1294	1.9991	2.1564	2.0605	1.7789
f_4	1	2.4522	2.3692	2.3097	2.4813	2.1044

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求解非线性方程的最优8阶史蒂芬森方法

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摘 要: 本文研究了非线性方程求根问题. 利用权函数方法, 获得了一种三步8阶收敛的史蒂芬森型方法. 实验结果表明本文提出的方法计算时间少于其它同阶的最优方法.

关 键 词: 史蒂芬森法; 无导数; 8阶收敛

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