

SOME IDENTITIES RELATED TO THE DEDEKIND SUMS

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Abstract: In this paper, we study a new mean value problem related to the Dedekind sums. By using the properties of character sum and the analytic method, we get two interesting mean value formulae for it.

Keywords: the Dedekind sum; mean value formula; identity

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1 Introduction

For a positive integer k and an arbitrary integer h , the classical Dedekind sums $S(h, k)$ is defined by

$$S(h, k) = \sum_{a=1}^k ((\frac{a}{k})) ((\frac{ah}{k})),$$

where

$$((x)) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \text{ is not an integer;} \\ 0, & \text{if } x \text{ is an integer.} \end{cases}$$

The various properties of $S(h, k)$ were investigated by many authors, see [2–4, 7–9]. For example, Carlitz [3] obtained a reciprocity theorem of $S(h, k)$. Conrey et al. [4] studied the mean value distribution of $S(h, k)$, and proved the following important and interesting asymptotic formula

$$\sum_{h=1}^k |S(h, k)|^{2m} = f_m(k) (\frac{k}{12})^{2m} + O((k^{\frac{9}{5}} + k^{2m-1+\frac{1}{m+1}}) \log^3 k), \quad (1.1)$$

where $\sum'_h$ denotes the summation over all h such that $(k, h) = 1$, and

$$\sum_{n=1}^{\infty} \frac{f_m(n)}{n^s} = 2 \frac{\zeta^2(2m)}{\zeta(4m)} \cdot \frac{\zeta(s+4m-1)}{\zeta^2(s+2m)} \zeta(s),$$

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$\zeta(s)$ is the Riemann zeta-function.

Jia [7] improved the error term in (1.1) as $O(k^{2m-1})$, provide $m \geq 2$. Zhang [9] improved the error term of (1.1) for $m = 1$. That is, he proved the following asymptotic formula

$$\sum_{h=1}^k {}' |S(h, k)|^2 = \frac{5}{144} k \phi(k) \cdot \frac{\prod_{p^\alpha \| k} ((1 + \frac{1}{p})^2 - \frac{1}{p^{3\alpha+1}})}{\prod_{p|k} (1 + \frac{1}{p} + \frac{1}{p^2})} + O(k \exp(\frac{4 \ln k}{\ln \ln k})),$$

where $p^\alpha \| k$ denotes that $p^\alpha | k$ and $p^{\alpha+1} \nmid k$ and $\exp(y) = e^y$.

Liu and Zhang [11] studied the hybrid mean value involving Dedekind sums and Kloosterman sums $K(m, n; q)$, which defined as follows (see [5] and [6]):

$$K(m, n; q) = \sum_{b=1}^q {}' e(\frac{mb + n\bar{b}}{q}),$$

where $e(y) = e^{2\pi i y}$, $\bar{b}$ denotes the solution of the equation $x \cdot b \equiv 1 \pmod{q}$. They proved the following conclusion:

Let q be a square-full number (i.e., $p | q$ if and only if $p^2 | q$), then we have

$$\sum_{a=1}^q {}' \sum_{b=1}^q {}' K(m, a; q) \overline{K(m, b; q)} S(a\bar{b}, q) = \frac{1}{12} \cdot q \cdot \phi^2(q) \prod_{p|q} (1 + \frac{1}{p}),$$

where $\sum_{a=1}^q {}'$ denotes the summation over all $1 \leq a \leq q$ such that $(a, q) = 1$, $\prod_{p|q}$ denotes the product over all distinct prime divisors p of q , $\phi(q)$ is the Euler function, and $\overline{f(n)}$ denotes the complex conjugation of $f(n)$.

In this paper, we use the analytic methods and mean value theorem of Dirichlet L -functions to study the hybrid mean value properties involving Dedekind sums and Legendre's symbol, and prove some new identities and asymptotic formulae. That is, we shall prove the following several conclusions:

Theorem 1.1 Let p be an odd prime with $p \equiv 3 \pmod{4}$, then we have the identity

$$\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} (\frac{a+1}{p})(\frac{b+1}{p}) S(a\bar{b}, p) = \frac{(p-1)(p-2)}{12} - h_p^2,$$

where h_p denotes the class number of the quadratic field $\mathbf{Q}(\sqrt{-p})$.

Theorem 1.2 Let p be an odd prime with $p \equiv 1 \pmod{4}$, then for any positive integer k , we have the identity

$$\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} (\frac{a+1}{p})(\frac{b+1}{p}) S^{2k-1}(a\bar{b}, p) = \frac{p \cdot (p-1)^{2k-1} \cdot (p-2)^{2k-1}}{(12p)^{2k-1}}.$$

Theorem 1.3 Let p be an odd prime, then we have the asymptotic formulae:

$$(A) \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \left(\frac{a+1}{p}\right) \left(\frac{b+1}{p}\right) S^2(a\bar{b}, p) = \frac{p^2 \cdot (p-11)}{144} + O(p \cdot \exp(\frac{4 \ln p}{\ln \ln p})), \text{ if } p \equiv 3 \pmod{4}.$$

$$(B) \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \left(\frac{a+1}{p}\right) \left(\frac{b+1}{p}\right) S^2(a\bar{b}, p) = \frac{p^2 \cdot (p-11-C_1)}{144} + O(p \cdot \exp(\frac{4 \ln p}{\ln \ln p})), \text{ if } p \equiv 1 \pmod{4}, \text{ where}$$

$C_1 = \frac{4\pi^4}{5} \prod_q^* \frac{(q^2+1)^2}{(q^2-1)^2}$ is a constant, $\prod_q^*$ denotes the product over all primes q such that $(\frac{q}{p}) = 1$, and $\exp(y) = e^y$.

As an application of Theorem 1.1 (of course, one can also give a proof directly), we can give an interesting computational formula for h_p , which we described as following:

Corollary 1.4 Let p be an odd prime with $p \equiv 3 \pmod{4}$, then we have the computational formula

$$h_p = \left\{ \sum_{a=1}^{p-1} S(a^2, p) \right\}^{\frac{1}{2}} = 2 \sum_{a=1}^{\frac{p-1}{2}} \left[\frac{a^2}{p} \right] - \frac{(p-1)(p-5)}{12} = \frac{p-1}{2} - \frac{2}{p} \sum_{i=1}^{\frac{p-1}{2}} r_i,$$

where $[x]$ denotes the greatest integer $\leq x$, r_i ($i = 1, 2, \dots, \frac{p-1}{2}$) denotes all quadratic residues mod p in the interval $[1, p-1]$.

2 Some Lemmas

In this section, we shall give some lemmas which are necessary in the proof of our theorems. First we have the following:

Lemma 2.1 Let $q > 2$ be an integer, then for any integer a with $(a, q) = 1$, we have the identity

$$S(a, q) = \frac{1}{\pi^2 q} \sum_{d|q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \pmod{d} \\ \chi(-1)=-1}} \chi(a) |L(1, \chi)|^2,$$

where $\phi(n)$ is the Euler function, $\sum_{\substack{\chi \pmod{d} \\ \chi(-1)=-1}}$ denotes the summation over all odd character

modulo d , $L(s, \chi)$ denotes the Dirichlet L -function corresponding to χ modulo d .

Lemma 2.2 Let p be an odd prime, then we have the asymptotic formulae

$$(I) \sum_{\substack{\chi \pmod{p} \\ \chi(-1)=-1}} |L(1, \chi)|^4 = \frac{5\pi^4}{144} \cdot p + O(\exp(\frac{4 \ln p}{\ln \ln p}));$$

$$(II) \sum_{\substack{\chi \pmod{p} \\ \chi(-1)=-1}} |L(1, \chi)|^2 \cdot |L(1, \chi\chi_2)|^2 = C_1 \cdot p + O(\exp(\frac{4 \ln p}{\ln \ln p})),$$

where $C_1 = \frac{\pi^4}{180} \prod_q^* \frac{(q^2+1)^2}{(q^2-1)^2}$ is a constant, $\prod_q^*$ denotes the product over all primes q such that $(\frac{q}{p}) = 1$.

Proof In fact Lemma 2.1 and Lemma 2.2 are two early results of the second author, their proof can be find in references [8] and [10].

Lemma 2.3 Let p be an odd prime, then for any non-real character $\chi \bmod p$, we have the identity $|\sum_{a=1}^{p-1} (\frac{a+1}{p}) \chi(a)| = \sqrt{p}$, where $(\frac{*}{p})$ is the Legendre's symbol.

Proof Since χ_2 is a primitive character mod p , so from the properties of Gauss sums $\tau(\chi)$ we have

$$\begin{aligned} \sum_{a=1}^{p-1} (\frac{a+1}{p}) \chi(a) &= \frac{1}{\tau(\chi_2)} \sum_{a=1}^{p-1} \chi(a) \sum_{b=1}^{p-1} \chi_2(b) e(\frac{b(a+1)}{p}) \\ &= \frac{1}{\tau(\chi_2)} \sum_{b=1}^{p-1} \chi_2(b) \bar{\chi}(b) e(\frac{b}{p}) \sum_{a=1}^{p-1} \chi(ab) e(\frac{ab}{p}) \\ &= \frac{1}{\tau(\chi_2)} \sum_{b=1}^{p-1} \chi_2(b) \bar{\chi}(b) e(\frac{b}{p}) \sum_{a=1}^{p-1} \chi(a) e(\frac{a}{p}) = \frac{\tau(\chi) \cdot \tau(\bar{\chi}\chi_2)}{\tau(\chi_2)}, \end{aligned} \quad (2.1)$$

where $\chi_2 = (\frac{*}{p})$ is the Legendre's symbol.

Now Lemma 2.3 follows from (2.1) and the identity $|\tau(\chi)| = \sqrt{p}$, if χ is not a principal character mod p .

3 Proof of Theorems

In this section, we shall complete the proof of our theorems. First we prove Theorem 1.1. For odd prime p , from Lemma 2.1 and the definition of $S(a, p)$ we have

$$S(a, p) = \frac{p}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(a) |L(1, \chi)|^2 \quad (3.1)$$

and

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1, \chi)|^2 = \frac{\pi^2(p-1)^2(p-2)}{12p^2}. \quad (3.2)$$

Then from (3.1) and Lemma 2.3 we have

$$\begin{aligned} &\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} (\frac{a+1}{p}) (\frac{b+1}{p}) S(ab, p) \\ &= \frac{p}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \left| \sum_{a=1}^{p-1} (\frac{a+1}{p}) \chi(a) \right|^2 \cdot |L(1, \chi)|^2 \\ &= \frac{p}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \left| \frac{\tau(\chi) \cdot \tau(\bar{\chi}\chi_2)}{\tau(\chi_2)} \right|^2 \cdot |L(1, \chi)|^2. \end{aligned} \quad (3.3)$$

If $p \equiv 3 \pmod{4}$, then $\chi_2(-1) = -1$, so for $\bar{\chi} = \chi_2$, $\bar{\chi}\chi_2$ is the principal character mod p and $|\tau(\bar{\chi}\chi_2)| = 1$. Note that $L(1, \chi_2) = \pi h_p / \sqrt{p}$, from (3.2), (3.3) and Lemma 2.3 we have

$$\begin{aligned} & \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \left(\frac{a+1}{p}\right) \left(\frac{b+1}{p}\right) S(a\bar{b}, p) = \frac{p}{\pi^2(p-1)} \left[\sum_{\substack{\chi \pmod{p} \\ \chi(-1)=-1}} p |L(1, \chi)|^2 - (p-1) |L(1, \chi_2)|^2 \right] \\ &= \frac{(p-1)(p-2)}{12} - \frac{p}{\pi^2} |L(1, \chi_2)|^2 = \frac{(p-1)(p-2)}{12} - h_p^2. \end{aligned}$$

This proves Theorem 1.1.

Now we prove Theorem 1.2. If $p \equiv 1 \pmod{4}$, then $\chi_2(-1) = 1$, so for any odd character χ , $\chi\chi_2$ is not the principal character mod p . This time, from the properties of Gauss sums, (3.1), (3.2) and Lemma 2.3 we have

$$\begin{aligned} & \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \left(\frac{a+1}{p}\right) \left(\frac{b+1}{p}\right) S^{2k-1}(a\bar{b}, p) = \left(\frac{p}{\pi^2(p-1)}\right)^{2k-1} \cdot p \cdot \left(\sum_{\substack{\chi \pmod{p} \\ \chi(-1)=-1}} |L(1, \chi)|^2 \right)^{2k-1} \\ &= \left(\frac{p}{\pi^2(p-1)}\right)^{2k-1} \cdot p \cdot \left(\frac{\pi^2(p-1)^2 \cdot (p-2)}{12p^2}\right)^{2k-1} = \frac{p \cdot (p-1)^{2k-1} \cdot (p-2)^{2k-1}}{(12p)^{2k-1}}. \end{aligned}$$

This proves Theorem 1.2.

To prove Theorem 1.3. First from (3.1) and Lemma 2.3 we have

$$\begin{aligned} & \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \left(\frac{a+1}{p}\right) \left(\frac{b+1}{p}\right) S^2(a\bar{b}, p) \\ &= \frac{p^2}{\pi^4(p-1)^2} \sum_{\substack{\chi \pmod{p} \\ \chi(-1)=-1}} \sum_{\substack{\lambda \pmod{p} \\ \lambda(-1)=-1}} \left| \sum_{a=1}^{p-1} \left(\frac{a+1}{p}\right) \chi(a) \lambda(a) \right|^2 \cdot |L(1, \chi)|^2 |L(1, \lambda)|^2 \\ &= \frac{p^2}{\pi^4(p-1)^2} \sum_{\substack{\chi \pmod{p} \\ \chi(-1)=-1}} \sum_{\substack{\lambda \pmod{p} \\ \lambda(-1)=-1}} \left| \frac{\tau(\chi\lambda) \cdot \tau(\bar{\chi}\bar{\lambda}\chi_2)}{\tau(\chi_2)} \right|^2 \cdot |L(1, \chi)|^2 \cdot |L(1, \lambda)|^2. \quad (3.4) \end{aligned}$$

If $p \equiv 3 \pmod{4}$, then χ_2 is an odd character mod p , and $\chi\lambda$ is an even character mod p . So $\bar{\chi}\bar{\lambda}\chi_2$ is not a principal character mod p . From (3.4), Lemma 2.2 and the properties of Gauss sums we have

$$\begin{aligned} & \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \left(\frac{a+1}{p}\right) \left(\frac{b+1}{p}\right) S^2(a\bar{b}, p) \\ &= \frac{p^2}{\pi^4(p-1)^2} \left[p \cdot \left(\sum_{\substack{\chi \pmod{p} \\ \chi(-1)=-1}} |L(1, \chi)|^2 \right)^2 - \sum_{\substack{\chi \pmod{p} \\ \chi(-1)=-1}} (p-1) |L(1, \chi)|^4 \right] \\ &= \frac{(p-1)^2(p-2)^2}{144p} - \frac{p^2}{\pi^4(p-1)} \sum_{\substack{\chi \pmod{p} \\ \chi(-1)=-1}} |L(1, \chi)|^4 = \frac{p^2 \cdot (p-11)}{144} + O(p \cdot \exp(\frac{4 \ln p}{\ln \ln p})). \end{aligned} \quad (3.5)$$

If $p \equiv 1 \pmod{4}$, then χ_2 is an even character mod p , this time we have

$$\begin{aligned}
& \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \left(\frac{a+1}{p}\right) \left(\frac{b+1}{p}\right) S^2(a\bar{b}, p) \\
&= \frac{p^2}{\pi^4(p-1)^2} \left[p \cdot \left(\sum_{\substack{\chi \pmod{p} \\ \chi(-1)=-1}} |L(1, \chi)|^2 \right)^2 - (p-1) \sum_{\substack{\chi \pmod{p} \\ \chi(-1)=-1}} |L(1, \chi)|^4 \right] \\
&\quad - \frac{p^2}{\pi^4(p-1)} \sum_{\substack{\chi \pmod{p} \\ \chi(-1)=-1}} |L(1, \chi)|^2 \cdot |L(1, \chi\chi_2)|^2 \\
&= \frac{p^2 \cdot (p-11)}{144} - C \cdot p^2 + O(p \cdot \exp(\frac{4 \ln p}{\ln \ln p})), \tag{3.6}
\end{aligned}$$

where $C = \frac{\pi^4}{180} \prod_q^* \frac{(q^2+1)^2}{(q^2-1)^2}$ is a constant, $\prod_q^*$ denotes the product over all primes q such that $(\frac{q}{p}) = 1$.

Now Theorem 1.3 follows from asymptotic formulae (3.5) and (3.6).

Using Lemma 2.1 we can also give a direct proof of Corollary 1.4. In fact if $p \equiv 3 \pmod{4}$, then $(\frac{-1}{p}) = -1$, so from Lemma 2.1 and note that the orthogonality of characters mod p , we have

$$\begin{aligned}
\sum_{a=1}^{p-1} S(a^2, p) &= \frac{p}{\pi^2(p-1)} \sum_{\substack{\chi \pmod{p} \\ \chi(-1)=-1}} \sum_{a=1}^{p-1} \chi^2(a) |L(1, \chi)|^2 \\
&= \frac{p}{\pi^2(p-1)} \cdot (p-1) \cdot |L(1, \chi_2)|^2 = h_p^2. \tag{3.7}
\end{aligned}$$

On the other hand, note that $((-x)) = -((x))$ and the set $\{1^2, 2^2, \dots, \frac{(p-1)^2}{4}, -1^2, -2^2, \dots, -\frac{(p-1)^2}{4}\}$ is a reduced residue system mod p , so from the definition of $S(a, p)$ we have

$$\begin{aligned}
\sum_{a=1}^{p-1} S(a^2, p) &= \sum_{a=1}^{p-1} \left\{ \sum_{b=1}^{\frac{p-1}{2}} \left(\frac{b^2}{p}\right) \left(\frac{b^2 a^2}{p}\right) + \sum_{b=1}^{\frac{p-1}{2}} \left(\frac{-b^2}{p}\right) \left(\frac{-b^2 a^2}{p}\right) \right\} \\
&= 2 \sum_{b=1}^{\frac{p-1}{2}} \left(\frac{b^2}{p}\right) \sum_{a=1}^{p-1} \left(\frac{b^2 a^2}{p}\right) = 4 \left(\sum_{b=1}^{\frac{p-1}{2}} \left(\frac{b^2}{p}\right) \right)^2 \\
&= 4 \left(\sum_{b=1}^{\frac{p-1}{2}} \frac{b^2}{p} - \frac{1}{2} \sum_{b=1}^{\frac{p-1}{2}} 1 - \sum_{b=1}^{\frac{p-1}{2}} \left[\frac{b^2}{p}\right] \right)^2 = 4 \left(\frac{(p-1)(p-5)}{24} - \sum_{b=1}^{\frac{p-1}{2}} \left[\frac{b^2}{p}\right] \right)^2. \tag{3.8}
\end{aligned}$$

Combining (3.7) and (3.8) we may immediately deduce the identity

$$h_p = \left\{ \sum_{a=1}^{p-1} S(a^2, p) \right\}^{\frac{1}{2}} = 2 \sum_{a=1}^{\frac{p-1}{2}} \left[\frac{a^2}{p}\right] - \frac{(p-1)(p-5)}{12}.$$

This completes the proof of our conclusions.

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关于Dedekind和的一些恒等式

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摘要: 本文研究了关于Dedekind和的一个新的均值问题. 利用特征和的性质以及解析的方法, 获得了两个有趣的均值公式.

关键词: Dedekind 和; 均值公式; 恒等式

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