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SOME RESULTS ABOUT THE DIVIDEND-PENALTY IDENTITY

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Abstract: In this paper, a Markov renewal risk model with a constant dividend barrier is considered, the matrix form of systems of integro-differential equations is presented and the analytical solutions to these systems are derived. By the general solution of the integro-differential equation, the dividend-penalty identity is obtained, which generalizes the results of the ref.[1].

Keywords: dividend payments; dividend-penalty identity; integro-differential equations

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1 Introduction

Recently, there was a more and more interest in the issue of risk models with dividend strategies. Obviously, dividend strategies can reflect the surplus cash flows more realistically in a insurance portfolio. The theories developed are very valuable in the devising and managing of products with dividends. In this paper, we discuss the dividend-penalty identity problem in a Markov risk model, which governed by a Markov arrival claim process and allows for claim sizes to be correlated with inter-claim times. The purpose of this paper is to show how the dividend penalty identity can be obtained by general solution of the integro-differential equation.

Suppose $\{Z_n; n \geq 0\}$ is an irreducible discrete time Markov chain with state space $E = \{1, \cdots, m\}$ and transition matrix

$$\mathbf{P} = (p_{i,j})_{i,j=1}^{m}.$$

Let u be the initial capital, c the premium rate, X_j the size of the jth claim and $N(t)$ is the number of claims up to time t. $F(x)$ is the distribution of the claim X_j. The surplus process $\{R(t); t \geq 0\}$ is defined as

$$R(t) = u + ct - \sum_{j=1}^{N(t)} X_j. \tag{1}$$

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Let W_i denote the duration between the arrivals of the $(i-1)$th and the ith claim and $W_0 = X_0 = 0$ a.s., then

$$\begin{aligned} & P(W_{n+1} \le x, X_{n+1} \le y, Z_{n+1} = j|\ Z_n = i, (W_r, X_r, Z_r), 0 \le r \le n) \\ = \ & P(W_1 \le x, X_1 \le y, Z_1 = j|\ Z_0 = i) = (1 - e^{-\lambda_i x}) p_{ij} F_j(y). \end{aligned} \tag{2}$$

The model is enriched by the payment of dividends to the share-holders of the company, and the surplus is modified accordingly. When the surplus exceeds a constant barrier $b \ge u$, dividends are paid continuously so that the surplus stays at the level b until next claim occurs. Let $\{R_b(t); t \ge 0\}$ be the surplus process with initial surplus u under the constant barrier above.

Define T_b =inf$\{t \ge 0 : R_b(t) < 0\}$ to be the time of ruin. Let $\delta > 0$ be the force interest and $w(x, y)$, for $x, y \ge 0$, be non-negative valued of penalty function. Let $\mathbf{p}_i = \mathbf{p}(\cdot|Z_0 = i)$, define

$$\phi_i(u; b) = \mathbf{E}_i[e^{-\delta T_b} w(R(T_b-), |R(T_b)|) I(T_b < \infty) | R(0) = u] \tag{3}$$

to be the discounted penalty function (Gerber-Shiu function) at T_b given that the initial surplus is u, given that the initial environment is i. Denote that when $b = \infty$,

$$\phi_i(u; b) = \phi_i(u),$$

that is the Gerber-Shiu function without dividend barrier.

2 Main Results and Proof

The Gerber-Shiu discounted penalty function under the constant dividend barrier is associated with the discounted penalty function for the process without dividend strategy. Define $T = \inf\{t \ge 0 : R(t) < 0\}$ to be the time of ruin of the surplus process (1), and for $\delta \ge 0$,

$$\phi_i(u) = \mathbf{E}_i[e^{-\delta T} \omega(R(T-), |R(T)|) I(T < \infty) | R(0) = u], \quad u \ge 0,\ i \in E$$

to be the Gerber -Shiu discounted penalty function, given the initial surplus u and the initial state i. We denote by c the constant premium rate for the surplus process (1) without dividend strategy. Function $\phi_i(u)$ investigated in Albrecher and Boxma (2005), which satisfies the following integro-differential equation, for $i \in E$,

$$\begin{aligned} & c\phi_i^{'}(u) \\ = \ & (\lambda_i + \delta)\phi_i(u) - \lambda_i \sum_{j=1}^{m} p_{ij} \left(\int_0^u \phi_j(u-y) dF_j(y) + \int_u^\infty \omega(u, y-u) dF_j(y) \right). \end{aligned}$$

Let

$$\vec{\Phi}(u) = (\phi_1(u), \cdots, \phi_m(u))^\top,$$

then an integro-differential equation in matrix form for $\vec{\Phi}(u)$ is given by

$$\vec{\Phi}'(u) = \mathbf{H}_c\vec{\Phi}(u) + \int_0^u \mathbf{G}_c(x)\vec{\Phi}(u-x)dx + \vec{h}(u), 0 < u < \infty, \tag{4}$$

where

$$\mathbf{H}_c = \mathrm{diag}((\lambda_1+\delta)/c, \cdots, (\lambda_m + \delta/)c)$$

and

$$\mathbf{G}_c(x) = -\begin{pmatrix} \frac{\lambda_1}{c} & & \\ & \ddots & \\ & & \frac{\lambda_m}{c} \end{pmatrix} \mathbf{P} \begin{pmatrix} f_1(x) & & \\ & \ddots & \\ & & f_m(x) \end{pmatrix}$$

are $m \times m$ matrices, and $\vec{h}(u)$ is an m-dimensional vector which is given by

$$\vec{h}(u) = \int_u^\infty \mathbf{G}_c(x)\omega(u, x-u)\vec{\mathbf{1}}dx, \tag{5}$$

where $\vec{\mathbf{1}} = (1, \cdots, 1)^\top$ is an m-dimensional column vector. The corresponding homogenous integro-differential equation of (4) is

$$\vec{\Phi}'(u) = \mathbf{H}_c\vec{\Phi}(u) + \int_0^u \mathbf{G}_c(x)\vec{\Phi}(u-x)dx. \tag{6}$$

By Theorem 2.3.1 in Burton [2], we give the analytical expression for $\vec{\Phi}(u)$ in the following lemma.

Lemma 1 Let $\mathbf{v}(u) = (v_{i,j}(u))_{i,j=1}^m$ be the $m \times m$ matrix whose columns are particular solutions to (6) with $\mathbf{v}(0) = \mathbf{I}$, where $\mathbf{I}$ is the $m \times m$ identity matrix. The solutions to equation (4) is

$$\vec{\Phi}(u) = \mathbf{v}(u)\vec{\Phi}(0) + \int_0^u \mathbf{v}(u-x)\vec{h}(x)dx, 0 \le u < \infty,$$

where $\mathbf{v}(u)$, $\vec{\Phi}(0)$ was given by (3.6) and (4.4) in [7].

As for $\phi_i(u;b)$, by similar approach as in Liu et al. (2010), we can get the Gerber- Shiu function (3) under the constant dividend strategy, satisfying the following integro-differential equation

$$\begin{aligned} & c\phi_i'(u;b) \\ = \quad & (\lambda_i+\delta)\phi_i(u;b) - \lambda_i \sum_{j=1}^m p_{ij}\Big(\int_0^u \phi_j(u-x;b)dF_j(x) + \int_u^\infty w(u, x-u)dF_j(x)\Big) \end{aligned}$$

with boundary condition $\phi_i'(b;b) = 0$.

Let

$$\vec{\mathbf{\Phi}}'(u;b) = (\Phi_1'(u;b), \cdots, \phi_m'(u;b))^\top,$$

$dF_i(x) = f_i(x)dx$, then an integro-differential equation in matrix form for $\vec{\Phi}(u;b)$ is given by

$$\vec{\mathbf{\Phi}}'(u;b) = \mathbf{H}_c\vec{\Phi}(u;b) + \int_0^u \mathbf{G}_c(x)\vec{\mathbf{\Phi}}(u-x;b)dx + \vec{h}(u), 0 \le u \le b \tag{7}$$

with boundary condition

$$\vec{\mathbf{\Phi}}'(b;b) = \vec{\mathbf{0}},$$

and $\vec{\mathbf{1}} = (1, \cdots, 1)^\top$, $\vec{\mathbf{0}} = (0, \cdots, 0)^\top$ are $m \times 1$ vectors.

Again we apply Theorem 2.3.1 in Burton [2] to obtain the analytical expression for $\vec{\Phi}(u;b)$ as follows

$$\vec{\Phi}(u;b) = \mathbf{v}(u)\vec{\Phi}(0;b) + \int_0^u \mathbf{v}(u-x)\vec{h}(x)dx, 0 \le u < b.$$

Now restricting $\vec{\Phi}(u;b)$ in (4) to $0 \le u < b$, we have

$$\vec{\Phi}(u;b) - \mathbf{v}(u)\vec{\Phi}(0;b) = \vec{\Phi}(u) - \mathbf{v}(u)\vec{\Phi}(0),$$

then $\vec{\Phi}(u;b)$ in (7) can be rewritten as

$$\vec{\Phi}(u;b) = \mathbf{v}(u)\ [\vec{\Phi}(0;b) - \vec{\Phi}(0)] + \vec{\Phi}(u) = \mathbf{v}(u)\vec{k}(b) + \vec{\Phi}(u),\ 0 \le u < b, \tag{8}$$

where $\vec{k}(b) = \vec{\Phi}(0;b) - \vec{\Phi}(0)$.

This formula (8) is the so-called dividend-penalty identity for a general class of the Markov risk model. Note that in (8), the expected discounted penalty function $\vec{\Phi}(u;b)$ for the modified surplus processes with dividend strategy can be expressed as the summation of the expected discounted penalty function $\vec{\Phi}(u)$ for the corresponding process without dividend strategy applied and a vector which is the product of $\mathbf{v}(u)$, a matrix function of u, and $\vec{k}(b)$, a vector function of b.

When $m = 1$, the model reduces to the classical compound Poisson risk model, the expected discounted penalty function $\vec{\Phi}(u;b)$ in (8) simplifies to

$$\phi(u;b) = \phi(u) + v(u)k(b), 0 \le u < b,$$

which is equation (5.1) in Lin et al. (2003). Here $\phi(u)$ ($m_\infty(u)$ in their paper) is the expected discounted penalty function under the classical risk process with premium rate c, and the function $v(u)$ satisfies reduced integro-differential equation (7) and the constant $k(b)$ is determined in their paper. We extend the results in [1] and show the this identity can be obtained by the general solution of the integro-differential equation.

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关于红利-惩罚等式的相关结果

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摘要: 本文研究了在一类马氏相关更新风险模型中的红利-惩罚等式的问题. 推导了在常数红利边界下，折扣惩罚函数满足的方程，利用解微分-积分方程的方法, 更简洁的推出了红利-惩罚等式相关的结果, 推广了文献[1]的结论.

关键词: 红利派发；红利-惩罚等式; 折扣惩罚函数; 微分-积分方程

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