Vol. 34 (2014)
No. 1

数 学 杂 志
J. of Math. (PRC)

A PROJECTION-RELATED CONE VOLUME INEQUALITY

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Abstract: In this paper, we study a projection-related cone volumn inequality. By using gradient projection of convex function, we obtain a new cone volume inequality restricted to the origin-symmetric convex bodies in $\mathbb{R}^n$. The inequality promotes the solves of Schneider's projection problem.

Keywords: convex bodies; projection body; Schneider's projection problem; cone volume inequality

2010 MR Subject Classification: 52A40; 52A20

Document code: A **Article ID:** 0255-7797(2014)01-0085-06

1 Introduction

A convex body K (i.e., a compact, convex subset with nonempty interior) in Euclidean n-space $\mathbb{R}^n$, is determined by its support function, $h(K,\cdot): S^{n-1} \mapsto \mathbb{R}^n$, on the unit sphere S^{n-1}, where $h(K,u) = \max\{u \cdot x | x \in K\}$ and where $u \cdot x$ denotes the standard inner product of u and x. The projection body, ΠK, of K is the convex body whose support function, for $u \in S^{n-1}$, is given by $h(\Pi K, u) = \mathrm{vol}_{n-1}(K|u^{\perp})$, where vol_{n-1} denotes the $(n-1)$-dimensional volume and $K|u^{\perp}$ denotes the image of the orthogonal projection of K onto the codimension 1 subspace orthogonal to u.

Projection bodies were introduced by Minkowski at the beginning of the previous century in connection with Cauchy' s surface area formula. Since 1980s, projection bodies received considerable attention. An important unsolved problem regarding projection bodies is Schneider's projection problem (see [7]): what is the least upper bound, as K ranges over the class of origin-symmetric convex bodies in $\mathbb{R}^n$, of the affine-invariant ratio

$$\Big[V(\Pi K)/V(K)^{n-1}\big]^{\frac{1}{n}},$$

where V is used to denote the n-dimensional volume.

An effective tool to study Schneider's projection problem is the cone volume functional U introduced by Lutwak, Yang and Zhang [1]: if P is a convex polytope in $\mathbb{R}^n$

* **Received date:** 2013-06-15 **Accepted date:** 2013-10-09

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which contains the origin o in its interior, then define $U(P)$ through the formula $U(P)^n = \frac{1}{n^n}\sum\limits_{u_{i_1}\wedge\cdots\wedge u_{in}\neq 0} h_{i_1}\cdots h_{i_n}a_{i_1}\cdots a_{i_n}$, where $u_{i_1},\cdots,u_{i_N}$ are the outer normal unit vectors to the corresponding facets F_i of $F_1,\cdots,F_N$ of P, and the facet with outer normal vector u_i has area (i.e., $(n-1)$-dimensional volume) a_i and distance h_i from the origin.

Let $V_i = \frac{1}{n}h_i a_i$. Then V_i is the volume of the cone $\mathrm{conv}(o, F_i)$, and

$$U(P)^n = \sum_{u_{i_1}\wedge\cdots\wedge u_{in}\neq 0} V_{i_1}\cdots V_{i_n},$$

obviously the functional U is centro-affine invariant, i.e.,

$$U(\o P) = U(P),\ \forall \o \in SL(n),$$

since $V(P) = \frac{1}{n}\sum\limits_{i=1}^{N} a_i h_i$, it follows that $U(P)/V(P) \leq 1$.

By the way, we observe the cone volume functional U has strong connection with the cone measure: for every star-shaped body $K \subseteq \mathbb{R}^n$, the cone measure of a subset A and vertex o. The cone measure appears in the Gromov-Milman theorem [9] on the concentration of Lipschitz functions on uniformly convex bodies. In [8], Anor established the precise relation between the surface measure and cone measure on the sphere of l_p^n.

One fundamental, but still remaining open extremum problem, on the ratio of U to V is posed by Lutwak, Deane, and Zhang [1].

Conjecture If P is a convex polytope in $\mathbb{R}^n$ with its centroid at the origin, then

$$\frac{U(P)}{V(P)} \geq \frac{(n!)^{1/n}}{n}$$

with equality when and only when P is a parallelotope.

The first progress on LYZ's conjecture was due to He, Leng and Li [2]. They proved that the conjecture is true when restricted to the class of origin-symmetric convex polytopes.

Theorem 1.1 Suppose that $P \subseteq \mathbb{R}^n$ is an origin-symmetric convex polytope, then

$$\frac{U(P)}{V(P)} \geq \frac{(n!)^{1/n}}{n}$$

with equality when and only when P is a parallelotope.

Lutwak, Yang and Zhang presented a version of Schneider's conjecture that has an affirmative answer. They proved the following important theorems.

Theorem 1.2 Suppose K is an origin-symmetric convex polytope in $\mathbb{R}^n$, then

$$\frac{V(\Pi K)}{U(K)^{\frac{n}{2}}V(K)^{\frac{n}{2}-1}} \leq 2^n\left(\frac{n^n}{n!}\right)^{\frac{1}{2}}$$

with equality when and only when K is a parallelotope.

By Theorems 1.1 and 1.2, the following theorem is obtained.

Theorem 1.3 Suppose K is an origin-symmetric convex polytope in $\mathbb{R}^n$, then

$$\frac{V(\Pi K)}{U(K)^{n-1}} \leq 2^n \frac{n^{n-1}}{(n!)^{(n-1)/n}}$$

with equality when and only when K is a parallelotope.

Theorem 1.3 can be seen as a modified version of Schneider's projection conjecture.

This paper is devoted to the study of LYZ' s conjecture. We give another answer to the cone volume inequality in origin-symmetric convex bodies.

Theorem 1.4 Suppose that $P \subset \mathbb{R}^j \times \mathbb{R}^{n-j}$ is an origin-symmetric convex body with interior points. $V(P)$ is the volume of P, K is a convex cone, the vertex of which is at the origin, then $V(K) \leq \frac{j}{n} V(P)$.

2 The Proofs of Theorem

Definition 2.1 If $P \subset \mathbb{R}^j \times \mathbb{R}^{n-j}, 1 \leq j \leq n-1$, is an origin-symmetric convex body with interior points. Suppose (x^o, y^o) is the centroid of P, $x^o \in \mathbb{R}^j, y^o \in \mathbb{R}^{n-j}$, $D = K|\mathbb{R}^j$, $L \subset \mathbb{R}^n$ is a j-dimensional subspace, and $f(x) = \mathrm{vol}_{n-j}(P \bigcap (L^\perp + x)), x \in D$, u_x is the outer normal unit vector of x on ∂D. For convex cone K, the vertex of which is at the origin, then define the volume of K by

$$V(K) = \frac{1}{n} \int_{\partial D} \langle (x - x^o), u_x \rangle f(x) dS(x) = \frac{1}{n} \int_{\partial D} (x - x^o) f(x) d\mathbf{S}(x).$$

Theorem 2.1 Let $P \in \mathbb{R}^j \times \mathbb{R}^{n-j}, 1 \leq j \leq n-1$, be an origin-symmetric smooth convex body with interior points, $V(P)$ is the volume of P, then

$$V(K) \leq \frac{j}{n} V(P). \tag{2.1}$$

Proof Suppose $D = K|\mathbb{R}^j$, $L \subset \mathbb{R}^n$ is a j-dimensional subspace, and

$$f(x) = \mathrm{vol}_{n-j}(P \bigcap (L^\perp + x)), x \in D.$$

Let (x^o, y^o) is the centroid of $P, x^o \in \mathbb{R}^j, y^o \in \mathbb{R}^{n-j}, u_x$ is the outer normal unit vector of x on ∂D, then

$$V(K) = \frac{1}{n} \int_{\partial D} \langle (x - x^o), u_x \rangle f(x) dS(x) = \frac{1}{n} \int_{\partial D} (x - x^o) f(x) d\mathbf{S}(x).$$

According to the Gauss formula

$$\begin{aligned} V(K) &= \frac{1}{n} \int_{\partial D} \begin{pmatrix} (x_1 - x_1^o) f(x_1, \cdots, x_j) \\ \vdots \\ (x_j - x_j^o) f(x_1, \cdots, x_j) \end{pmatrix} d\mathbf{S}(x) \\ &= \frac{1}{n} \int_D [f_1' \cdot (x_1 - x_1^o) + \cdots + f_j' \cdot (x_j - x_j^o) + j \cdot f] d\sigma \\ &= \frac{j}{n} \int_D f d\sigma + \frac{1}{n} \int_D (\mathrm{grad} f, (x - x^o)) d\sigma \\ &= \frac{j}{n} V(P) + \frac{1}{n} \int_D (\mathrm{grad} f, (x - x^o)) d\sigma. \end{aligned}$$

The direction of grad$g$ varies with the direction of contour surface, and is directed from the low value to the high value, therefore the contour surface and $(x-x^o)$ form an obtuse angle, obviously $\int_D(\mathrm{grad}f,(x-x^o))d\sigma\leq 0$, so that $V(K)\leq \frac{j}{n}V(P)$.

Lemma 2.2 Let $P\subset\mathbb{R}^n$ be a convex body and $L\subset\mathbb{R}^n$ be a j-dimensional subspace, $1\leq j\leq n-1$. If $f:L\mapsto\mathbb{R}, f(x)=\mathrm{vol}_{n-j}(P\bigcap(L^{\perp}+x))$, then $f^{\frac{1}{n-j}}$ is concave on $P\mid L$.

From [6], we know that continuous convex function on a Banach space can be approximated by a smooth convex function, it follows immediately that

Lemma 2.3 Let $f^{\frac{1}{n-j}}$ be a concave function on D , for any $\varepsilon>0$, there exists a $M>0$ and a smooth concave function $g^{\frac{1}{n-j}}$, $\mid f^{\frac{1}{n-j}}-g^{\frac{1}{n-j}}\mid<\frac{\varepsilon}{M}$, then f approximates g.

Proof Suppose $\widetilde{f}=f^{\frac{1}{n-j}},\widetilde{g}=g^{\frac{1}{n-j}}$, from the Lagrange's mean value theorem, it follows that

$$\mid\widetilde{f}^{n-j}-\widetilde{g}^{n-j}\mid=\mid(n-j)u^{n-j-1}(\widetilde{f}-\widetilde{g})\mid<\frac{\varepsilon}{M},$$

let $f\leq u\leq g$ (or $g\leq u\leq f$), the function u is bounded, then $\mid f-g\mid=\mid\widetilde{f}^{n-j}-\widetilde{g}^{n-j}\mid<\varepsilon$, so that f approximates g.

Proof of Theorem 1.4 Suppose $D=K|\mathbb{R}^j$, $L\subset\mathbb{R}^n$ is a j-dimensional subspace, and $f(x)=\mathrm{vol}_{n-j}(P\bigcap(L^{\perp}+x)),x\in D$.

Let (x^o,y^o) is the centroid of P, $x^o\in\mathbb{R}^j,y^o\in\mathbb{R}^{n-j}$, u_x is the outer normal unit vector of x on ∂D, and

$$\begin{aligned}V(K)&=\frac{1}{n}\int_{\partial D}((x-x^o),u_x)f(x)dS(x)\\&=\frac{1}{n}\int_{\partial D}(x-x^o)|f-g+g|d\mathbf{S}(x)\\&\leq\frac{1}{n}\int_{\partial D}(x-x^o)|f-g|d\mathbf{S}(x)+\frac{1}{n}\int_{\partial D}(x-x^o)gd\mathbf{S}(x)\\&=\frac{1}{n}\int_{\partial D}(x-x^o)\varepsilon d\mathbf{S}(x)+\frac{1}{n}\int_{\partial D}(x-x^o)gd\mathbf{S}(x)\\&=\frac{1}{n}\int_{\partial D}(x-x^o)gd\mathbf{S}(x)+\frac{j}{n}\int_D\varepsilon d\sigma.\end{aligned}$$

According to the Gauss formula

$$\begin{aligned}V(K)&=\frac{1}{n}\int_{\partial D}\begin{pmatrix}(x_1-x_1^o)g(x_1,\cdots,x_j)\\\vdots\\(x_j-x_j^o)g(x_1,\cdots,x_j)\end{pmatrix}d\mathbf{S}(x)+\frac{j}{n}\int_D\varepsilon d\sigma\\&=\frac{1}{n}\int_D[g_1'\cdot(x_1-x_1^o)+\cdots+g_j'\cdot(x_j-x_j^o)+j\cdot g]d\sigma+\frac{j}{n}\int_D\varepsilon d\sigma\\&=\frac{j}{n}\int_D gd\sigma+\frac{1}{n}\int_D(\mathrm{grad}g,(x-x^o))d\sigma+\frac{j}{n}\int_D\varepsilon d\sigma\\&=\frac{j}{n}\int_D(g+\varepsilon)d\sigma+\frac{1}{n}\int_D(\mathrm{grad}g,(x-x^o))d\sigma.\end{aligned}$$

The direction of grad$g$ varies with the direction of contour surface, and is directed from the low value to the high value, therefore the contour lines and $(x - x^o)$ form an obtuse angle, obviously

$$\int_D (\mathrm{grad} g, (x - x^o)) d\sigma \le 0,$$

so that $V(K) \le \frac{j}{n} \int_D (g + \varepsilon) d\sigma$, since $|f - g| < \varepsilon$, then $V(K) \le \frac{j}{n} V(P)$.

This completes the proof.

For origin-symmetric convex polytopes, the first progress was due to He-Leng-Li [2]. They gave an affirmative answer to the LYZ' s conjecture in $\mathbb{R}^n$. They proved the inequality

$$\sum_{u_{i_1} \wedge \cdots \wedge u_{i_j} \wedge u_{i_k} = 0} V_{i_k} \le \frac{j}{n} V(P), 1 \le j \le n - 1.$$

And from the definition of $U(P)$, the proof of Theorem 1.1 was obtained in [2]. To make the paper self-contained, we present it here.

Proof of Theorem 1.1

$$\begin{aligned}
U(P)^n &= \sum_{u_{i_1} \wedge \cdots \wedge u_{in} \ne 0} V_{i_1} \cdots V_{i_n} \\
&= \sum_{u_{i_1} \wedge \cdots \wedge u_{i_{n-1}} \ne 0} V_{i_1} \cdots V_{i_{n-1}} (V - \sum_{u_{i_1} \wedge \cdots \wedge u_{i_{n-1}} \wedge u_{i_k} = 0} V_{i_k}) \\
&\ge \sum_{u_{i_1} \wedge \cdots \wedge u_{i_{n-1}} \ne 0} V_{i_1} \cdots V_{i_{n-1}} (V - \frac{n-1}{n} V) \\
&\vdots \\
&= \frac{(n-2)!}{n^{n-2}} V^{n-2} \sum_{u_{i_1} u_{i_2} \ne 0} V_{i_1} V_{i_2} \\
&= \frac{(n-2)!}{n^{n-2}} V^{n-2} \sum_{u_{i_1} \ne 0} V_{i_1} (V - \sum_{u_{i_1} u_{i_k} = 0} V_k) \\
&\ge \frac{(n-1)!}{n^{n-1}} V^{n-1} \sum_{u_{i_1} \ne 0} V_{i_1} = \frac{n!}{n^n} V^n,
\end{aligned}$$

that is,

$$U(P) \ge \frac{(n!)^{1/n}}{n} V(P),$$

where the equality holds when and only when P is a parallelotope.

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一个与投影体相关的锥体积不等式

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摘 要: 本文研究了一个与投影体相关的锥体积不等式. 利用凸函数的梯度性质, 获得了n 维欧氏空间中关于任意原点对称凸体的一个锥体积不等式, 推进了Schneider投影问题的解决.

关 键 词: 凸体; 投影体; Schneider投影问题; 锥体积不等式

MR(2010) **主题分类号:** 52A40; 52A20 **中图分类号:** O186.5