Vol. 34 ( 2014 )
No. 1

数 学 杂 志
J. of Math. (PRC)

# ORLICZ PROJECTION BODIES OF ELLIPSOIDS

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**Abstract:** In this paper, we study the problem of the Orlicz projection bodies recently introduced by Lutwak, Yang and Zhang. By using the linear invariant property of Orlicz projection bodies, we obtain the result that the Orlicz projection bodies of ellipsoids are still ellipsoids. As examples, we compute two concrete support functions of Orlicz projection bodies of the unit ball for two specific convex functions.

**Keywords:** convex body; support function; projection bodies; Orlicz projection bodies

**2010 MR Subject Classification:** 52A20

**Document code:** A **Article ID:** 0255-7797(2014)01-0079-06

## 1 Introduction

In a series of ground-breaking work by Lutwak, Yang and Zhang [7, 8], the classical Brunn-Minkowski theory emerged at the turn of the 19th into the 20th century and then the $L_p$ Brunn-Minkowski theory originated from Lutwak's seminal work [4, 5], were remarkably generalized to the more broad framework, which is the so-called Orlicz-Brunn-Minkowski theory.

Within the Orlicz-Brunn-Minkowski theory, Orlicz projection body is spontaneously the important object. In retrospect, the two classical inequalities which connect the volume of a convex body with that of its polar projection body are the Petty and Zhang projection inequalities. The Petty projection inequality led to the affine Sobolev inequality [9] that is stronger than the classical Sobolev inequality and yet is independent of any underlying Euclidean structure. The $L_p$ analogue of projection bodies and the celebrated Petty projection inequality was established in [1] by Lutwak, Yang and Zhang, and independently derived by Campi and Gronchi [2] using an alternate approach. Recently, Lutwak, Yang and Zhang [7] established the corresponding Orlicz version.

We consider convex $\phi : \mathbb{R} \to [0, \infty)$ such that $\phi(0) = 0$. This means that $\phi$ must be decreasing on $(-\infty, 0]$ and increasing on $[0, \infty)$. We will assume throughout that one of these is happening strictly so; i.e., $\phi$ is either strictly decreasing on $(-\infty, 0]$ or strictly increasing on $[0, \infty)$. The class of such $\phi$ will be denoted by $\mathcal{C}$.

* **Received date:** 2012-03-05 **Accepted date:** 2012-06-20

**Foundation item:** Supported by National Natural Science Foundation of China (11001163); Innovation Program of Shanghai Municipal Education Commission (11YZ11); Shanghai Leading Academic Discipline Project(J50101).

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Let $K$ be a convex body in $\mathbb{R}^n$ that contains the origin in its interior and has volume $|K|$. For $\phi \in \mathcal{C}$, the Orlicz projection body $\Pi_\phi K$ of $K$ is defined as the body whose support function is given by

$$h_{\Pi_\phi K}(x) = \inf \left\{ \lambda > 0 : \int_{\partial K} \phi\left( \frac{x \cdot v(y)}{\lambda y \cdot v(y)} \right) y \cdot v(y) d\mathcal{H}^{n-1}(y) \leq n|K| \right\},$$

where $v(y)$ is the outer unit normal of $\partial K$ at $y \in \partial K$, where $x \cdot v(y)$ denotes the inner product of $x$ and $v(y)$, and $\mathcal{H}^{n-1}$ is $(n-1)$-dimensional Hausdorff measure.

With $\phi_1(t) = |t|$, it turns out that for $u \in S^{n-1}$, $h_{\Pi_{\phi_1} K}(u) = \frac{c_n}{|K|}|K_u|$, where $|K_u|$ denotes the $(n-1)$-dimensional volume of $K_u$, the image of the orthogonal projection of $K$ onto the subspace $u^\perp$. Thus $\Pi_{\phi_1} K = \frac{c_n}{|K|}\Pi K$, where $\Pi K$ is the classical projection body of $K$ introduced by Minkowski.

With $\phi_p(t) = |t|^p$, and $p \geq 1$, $\Pi_{\phi_p} K = \frac{c_{n,p}}{|K|^{\frac{1}{p}}}\Pi_p K$, where $\Pi_p K$ is the $L_p$ projection body of $K$, defined as the convex body whose support function is given by

$$h_{\Pi_p K}(x) = \left\{ \int_{\partial K} |x \cdot v(y)|^p |y \cdot v(y)|^{1-p} d\mathcal{H}^{n-1}(y) \right\}^{1/p}.$$

In this paper, we demonstrate the fact that the Orlicz projection bodies of ellipsoids are still ellipsoids in $\mathbb{R}^n$. As examples, we compute two concrete support functions of Orlicz projection bodies of the unit ball for two specific convex functions.

## 2 Basics Regarding Convex Bodies

The setting for this paper is the $n$-dimensional Euclidean space $\mathbb{R}^n$. We write $e_1, \cdots, e_n$ for the standard orthonormal basis of $\mathbb{R}^n$. Throughout this paper, $B^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$ denotes the unit ball centered at the origin, and $\omega_n = |B^n|$ denotes its $n$-dimensional volume.

A convex body is a compact convex subset of $\mathbb{R}^n$ with nonempty interior. All the convex bodies of $\mathbb{R}^n$ will be denoted by $\mathcal{K}_0^n$. Associated with a convex body $K$ is its support function $h_K$ defined on $\mathbb{R}^n$ by $h_K(x) = \max\{x \cdot y : y \in K\}$. Thus, if $y \in \partial K$, then $h_K(v_K(y)) = v_K(y) \cdot y$, where $v_K(y)$ denotes an outer unit normal to $\partial K$ at $y$.

For more detailed facts on convex bodies, you can refer the excellent books authored by Schneider [3] and Gardner [6].

## 3 Main Results

In [7], it is proved that the definition of $h_{\Pi_\phi K}(x)$ is equivalent to the following definition:

$$h_{\Pi_\phi K}(x) = \inf \left\{ \lambda > 0 : \int_{S^{n-1}} \phi\left( \frac{x \cdot u}{\lambda h_K(u)} \right) h_K(u) dS_K(u) \leq n|K| \right\},$$

or equivalently,

$$h_{\Pi_\phi K}(x) = \inf \left\{ \lambda > 0 : \int_{S^{n-1}} \phi\left( \frac{1}{\lambda}(x \cdot u)\rho_{K^*}(u) \right) dV_K(u) \leq 1 \right\}.$$

The polar body of $\Pi_\phi K$ will be denoted by $\Pi_\phi^* K$.

Since the area measure $S_K$ cannot be concentrated on a closed hemisphere of $S^{n-1}$, and since we assume that $\phi$ is strictly increasing on $[0,\infty)$ or strictly decreasing on $(-\infty,0]$, it follows that the function

$$\lambda \longmapsto \int_{S^{n-1}} \phi\left(\frac{1}{\lambda}(x\cdot u)\rho_{K^*}(u)\right) dV_K(u)$$

is strictly decreasing in $(0,\infty)$. Thus we have

**Lemma 1** Suppose $\phi\in\mathcal{C}$, and $K\in\mathcal{K}_0^n$. If $x_0\in\mathbb{R}^n\setminus\{0\}$, then

$$\int_{S^{n-1}} \phi\left(\frac{x_0\cdot v}{\lambda_0 h_K(v)}\right) dV_K(v) > 1,$$

$=1$, or $<1$, respectively, if and only if $h_{\Pi_\phi K}(x_0) > \lambda_0$, $=\lambda_0$, or $<\lambda_0$, respectively.

From Lemma 1, we can show immediately the inclusion relation, which is a monotonicity of Orlicz projection body in some sense.

**Theorem 1** If $K\in\mathcal{K}$ and $\phi_1,\phi_2\in\mathcal{C}$, $\phi_1\le\phi_2$, then $\Pi_{\phi_1}K\subseteq\Pi_{\phi_2}K$.

**Proof** $\forall u\in S^{n-1}$, let $h_{\Pi_{\phi_1}K}(u)=\lambda$. In terms of Lemma 1, it has

$$\frac{1}{n|K|}\int_{S^{n-1}} \phi_1\left(\frac{u\cdot v}{\lambda h_K(v)}\right) h_K(v)dS_K(v) = 1.$$

Since $\phi_1\le\phi_2$, we have

$$\frac{1}{n|K|}\int_{S^{n-1}} \phi_2\left(\frac{u\cdot v}{\lambda h_K(v)}\right) h_K(v)dS_K(v) \ge 1$$

from Lemma 1 again, we have $h_{\Pi_{\phi_2}K}(u)\ge\lambda$. Therefore $h_{\Pi_{\phi_1}K}(u)\le h_{\Pi_{\phi_2}K}(u)$, that is $\Pi_{\phi_1}K\subseteq\Pi_{\phi_2}K$. This completes the proof.

**Theorem 2** The Orlicz projection body $\Pi_\phi E$ of the ellipsoid $E$ is still an ellipsoid.

**Proof** First, we prove that the Orlicz projection body of the unit ball is still a ball centered at the origin. Suppose $A\in SO(n)$ and $u\in S^{n-1}$, let $z=A^t v$. Then

$$\frac{1}{\omega_n}\int_{B^n}\phi(\frac{1}{\lambda}Au\cdot v)dv = \frac{1}{\omega_n}\int_{B^n}\phi(\frac{1}{\lambda}u\cdot A^t v)dA^t v = \frac{1}{\omega_n}\int_{A^tB^n}\phi(\frac{1}{\lambda}u\cdot z)dz$$
$$=\frac{1}{\omega_n}\int_{B^n}\phi(\frac{1}{\lambda}u\cdot z)dz.$$

It yields $h_{\Pi_\phi B^n}(Au)=h_{\Pi_\phi B^n}(u)$, which implies that the Orlicz projection body of the unit ball is still a ball.

Second, suppose the ellipsoid $E=AB^n$, $A\in GL(n)$. According to Lemma 2.6 in [7], we have

$$\Pi_\phi(E)=\Pi_\phi(AB^n)=A^{-t}\Pi_\phi(B^n).$$

In view of the just verified fact, it can conclude that $\Pi_\phi E$ is still an ellipsoid.

This completes the proof.

In the following, we compute the support functions of Orlicz projection bodies of the unit ball, when $\phi_1(x) = e^{|x|} - 1$ and $\phi_2(x) = e^{x^2} - 1$, respectively. Obviously, $\phi_1$ and $\phi_2$ are both belong to the class $\mathcal{C}$.

(1) $\phi_1(x) = e^{|x|} - 1$. We know that

$$\begin{aligned} h_{\Pi_{\phi_1} B^n}(e_1) &= \inf\{\lambda > 0 : \int_{S^{n-1}} \phi_1(\frac{1}{\lambda} e_1 \cdot u) dS(u) \le n\omega_n\} \\ &= \inf\{\lambda > 0 : \int_{S^{n-1}} (e^{\frac{|u_1|}{\lambda}} - 1) dS(u) \le n\omega_n\}. \end{aligned}$$

From Lemma 1, we have

$$\int_{S^{n-1}} (e^{\frac{|u_1|}{\lambda_0}} - 1) dS(u) = n\omega_n \Longleftrightarrow h_{\Pi_{\phi_1} B^n}(e_1) = \lambda_0.$$

So, it has

$$\begin{aligned} &\int_{S^{n-1}} e^{\frac{|u_1|}{\lambda_0}} dS(u) = 2n\omega_n \\ \Longrightarrow &\omega_{n-1} \int_{-1}^{1} e^{\frac{|u_1|}{\lambda_0}} (1 - u_1^2)^{\frac{n-3}{2}} du_1 = 2n\omega_n \\ \Longrightarrow &\int_0^1 e^{\frac{u_1}{\lambda_0}} (1 - u_1^2)^{\frac{n-3}{2}} du_1 = \frac{n\omega_n}{\omega_{n-1}} \\ \Longrightarrow &\sum_{k=0}^{\infty} \frac{1}{k! \lambda_0^k} \int_0^1 u_1^k (1 - u_1^2)^{\frac{n-3}{2}} du_1 = \frac{n\omega_n}{\omega_{n-1}}. \end{aligned}$$

Let $u_1^2 = t$, it gives

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{1}{2k! \lambda_0^k} \beta(\frac{k+1}{2}, \frac{n-1}{2}) = \frac{n\omega_n}{\omega_{n-1}} \\ \Longrightarrow &\sum_{k=0}^{\infty} \frac{1}{2k! \lambda_0^k} \frac{\Gamma(\frac{k+1}{2})\Gamma(\frac{n-1}{2})}{\Gamma(\frac{k+n}{2})} = \frac{n\omega_n}{\omega_{n-1}} \\ \Longrightarrow &\sum_{k=0}^{\infty} \frac{1}{k! \lambda_0^k} \frac{\Gamma(\frac{k}{2} + \frac{1}{2})}{\Gamma(\frac{k}{2} + \frac{n}{2})} = \frac{n(n-1)\pi^{\frac{1}{2}}}{\Gamma(1 + \frac{n}{2})}, \end{aligned} \tag{1}$$

which gives the required formula which $h_{\Pi_{\phi_1} B^n}$ is satisfied.

(2) $\phi_2(x) = e^{x^2} - 1$. We know that

$$\begin{aligned} h_{\Pi_{\phi_2} B^n}(e_1) &= \inf\{\lambda > 0 : \int_{S^{n-1}} \phi_2(\frac{1}{\lambda} e_1 \cdot u) dS(u) \le n\omega_n\} \\ &= \inf\{\lambda > 0 : \int_{S^{n-1}} (e^{\frac{u_1^2}{\lambda^2}} - 1) dS(u) \le n\omega_n\}. \end{aligned}$$

From Lemma 1, we have

$$\int_{S^{n-1}} (e^{\frac{u_1^2}{\lambda_1^2}} - 1) dS(u) = n\omega_n \Longleftrightarrow h_{\Pi_{\phi_2} B^n}(e_1) = \lambda_1.$$

So, it has

$$
\begin{aligned}
&\int_{S^{n-1}} e^{\frac{u_1^2}{\lambda_1^2}} dS(u) = 2n\omega_n \\
\Longrightarrow &\omega_{n-1} \int_{-1}^{1} e^{\frac{u_1^2}{\lambda_1^2}} (1-u_1^2)^{\frac{n-3}{2}} du_1 = 2n\omega_n \\
\Longrightarrow &\int_0^1 e^{\frac{u_1^2}{\lambda_1^2}} (1-u_1^2)^{\frac{n-3}{2}} du_1 = \frac{n\omega_n}{\omega_{n-1}} \\
\Longrightarrow &\sum_{k=0}^{\infty} \frac{1}{k!\lambda_1^{2k}} \int_0^1 u_1^{2k}(1-u_1^2)^{\frac{n-3}{2}} du_1 = \frac{n\omega_n}{\omega_{n-1}}.
\end{aligned}
$$

Let $u_1^2 = t$, it gives

$$
\begin{aligned}
&\sum_{k=0}^{\infty} \frac{1}{2k!\lambda_1^{2k}} \beta(k+\frac{1}{2}, \frac{n-1}{2}) = \frac{n\omega_n}{\omega_{n-1}} \\
\Longrightarrow &\sum_{k=0}^{\infty} \frac{1}{2k!\lambda_1^{2k}} \frac{\Gamma(k+\frac{1}{2})\Gamma(\frac{n-1}{2})}{\Gamma(k+\frac{n}{2})} = \frac{n\omega_n}{\omega_{n-1}} \\
\Longrightarrow &\sum_{k=0}^{\infty} \frac{1}{k!\lambda_1^{2k}} \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+\frac{n}{2})} = \frac{n(n-1)\pi^{\frac{1}{2}}}{\Gamma(1+\frac{n}{2})},
\end{aligned} \tag{2}
$$

which gives the required formula which $h_{\Pi_{\phi_2} B^n}$ is satisfied.

In view of the very similarity between formulas (1) and (2), we set out to compare which number is more larger between $\lambda_0$ and $\lambda_1$. For this aim, we consider the monotonicity of the following function $f(x)$.

**Lemma 2** The function $f(x) = \frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+\frac{n}{2}+1)}$ is strictly decreasing on $[0,\infty)$ with respect to $x$.

**Proof** Constructing a function $F(x) = \ln \frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+\frac{n}{2}+1)} = \ln\Gamma(x+\frac{1}{2}) - \ln\Gamma(x+\frac{n}{2}+1)$. Since function $\Gamma(x)$ is infinitely differentiable, we have

$$
\Gamma(x) = \lim_{m\to\infty} \frac{m!m^x}{(m+x)(m-1+x)\cdots(1+x)x}, \ \forall x \in (0,\infty),
$$

then

$$
\Gamma(x+\frac{1}{2}) = \lim_{m\to\infty} \frac{m!m^{x+\frac{1}{2}}}{(m+x+\frac{1}{2})(m-1+x+\frac{1}{2})\cdots(1+x+\frac{1}{2})(x+\frac{1}{2})}.
$$

By the above expansion and the continuity of the natural logarithmic function, $\ln\Gamma(x+\frac{1}{2})$ can be written as

$$
\ln\Gamma(x+\frac{1}{2}) = \lim_{m\to\infty} \Big(\ln m! + (x+\frac{1}{2})\ln m - \sum_{j=0}^{m} \ln(x+\frac{1}{2}+j)\Big).
$$

Since this sequence is absolutely convergent, we may interchange differentiation and limits

$$
\frac{d}{dx}\Big(\ln\Gamma(x+\frac{1}{2})\Big) = \lim_{m\to\infty} \Big(\ln m - \sum_{j=0}^{m} \frac{1}{x+\frac{1}{2}+j}\Big).
$$

It gives

$$F'(x) = \lim_{m\to\infty}\sum_{j=0}^{m}(\frac{1}{x+\frac{n}{2}+1+j} - \frac{1}{x+\frac{1}{2}+j}) < 0 \ \ (n \geq 1, n \in Z).$$

We obtain that $f(x) = \frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+\frac{n}{2}+1)}$ is strictly decreasing on $[0,\infty)$ in terms with $x$. This completes the proof.

Hence, according to Lemma 2, we obtain the following results $\lambda_0 > \lambda_1^2$, or equivalently,

$$h_{\Pi_{\phi_1}B^n}(e_1) > h^2_{\Pi_{\phi_2}B^n}(e_1),$$

where $\phi_1(x) = e^{|x|} - 1$, $\phi_2(x) = e^{x^2} - 1$, that is

$$\frac{V(\Pi_{\phi_1}B^n)}{V(B^n)} > \left(\frac{V(\Pi_{\phi_2}B^n)}{V(B^n)}\right)^2.$$

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# 椭球的 Orlicz 投影体

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**摘 要**: 本文研究了文献[1]所引入的 Orlicz 投影体问题. 利用 Orlicz 投影体在线性变换下的不变性, 获得了椭球的 Orlicz 投影体仍是椭球的结果. 作为例子, 计算了当取两个特定的凸函数时单位球的 Orlicz 投影体的支持函数.

**关键词**: 凸体; 支持函数; 投影体; Orlicz 投影体

MR(2010)**主题分类号**: 52A20 **中图分类号**: O186.5