Vol. 34 (2014)
No. 1

数 学 杂 志
J. of Math. (PRC)

BOUNDEDNESS OF OPERATOR CONVEXITY FOR CONVEX FUNCTIONS AND APPLICATIONS FOR OPERATOR MEANS

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Abstract: In this paper, we estimate boundedness of operator convexity for convex functions. As an application, we obtain some relations between the power of operator means (arithmetic mean, geometric mean, chaotically geometric mean) and those means of operator powers. In particular, we obtain the order relation between arithmetic mean and chaotically geometric mean.

Keywords: operator convex; arithmetic mean; geometric mean; chaotically geometric mean

2010 MR Subject Classification: 47A56; 47L25

Document code: A **Article ID:** 0255-7797(2014)01-0065-07

1 Preliminary

Throughout this paper, a capital letter means a bounded linear operator on Hilbert space H. An operator A is called positive, in symbol, $A \geq 0$ if $(Ax, x) \geq 0$ for all $x \in H$. T is called strictly positive (simply $T > 0$) if T is positive and invertible, α_i are positive numbers with $\sum\limits_{i=1}^{n} \alpha_i = 1$.

A continuous function f on interval I is called operator convex on I, if $\sigma(A)$, $\sigma(B) \subset I$,

$$f((1-\alpha)A + \alpha B) \leq (1-\alpha)f(A) + \alpha f(B) \quad \text{holds for} \quad \alpha \in [0,1]. \tag{1.1}$$

Convex functions and operator convex functions are different. Typical example of such function is t^r on $(0, \infty)$, which is a convex function for $r > 2$, but is not operator convex.

In [1], Ando, Li and Mathias proposed a definition of the geometric mean for an n-tuple of positive operators and showed that it has many required properties on the geometric mean. Following [3, 4], we recall the definition of the weighted geometric mean $G[n,t]$ with $t \in [0,1]$ for an n-tuple of positive invertible operators $A_1, A_2, \cdots, A_n$. Let $G[2,t](A_1, A_2) = A_1 \sharp_t A_2$. For $n \geq 3$, $G[n,t]$ is defined inductively as follows: put $A_i^{(0)} = A_i$ for all $i = 1, 2, \cdots, n$, and

$$A_i^{(r)} = G[n-1,t]((A_j^{(r-1)})_{j\neq i}) = G[n-1,t](A_1^{(r-1)}, \cdots, A_{i-1}^{(r-1)}, A_{i+1}^{(r-1)}, \cdots, A_n^{(r-1)})$$

* **Received date:** 2013-02-08 **Accepted date:** 2013-05-07

Foundation item: Partially Supported by NSFC (10471085); the Basic Science and Technological Frontier Project of Henan Province (132300410261).

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inductively for r. Then the sequence $\{A_i^{(r)}\}_{r=0}^{\infty}$ have the same limit for all $i = 1, 2, \cdots, n$ in the Thompson metric. So we can define

$$G[n,t](A_1, A_2, \cdots, A_n) = \lim_{r\to\infty} A_i^{(r)}.$$

Similarly, we can define the weighted arithmetic mean as follows: Let $A[2,t](A_1, A_2) = (1-t)A_1 + tA_2$. For $n \geq 3$, put $\tilde{A}_i^{(0)} = A_i$ for all $i = 1, 2, \cdots, n$ and

$$\tilde{A}_i^{(r)} = A[n-1,t]((\tilde{A}_j^{(r-1)})_{j\neq i}) = A[n-1,t](\tilde{A}_1^{(r-1)}, \cdots, \tilde{A}_{i-1}^{(r-1)}, \tilde{A}_{i+1}^{(r-1)}, \cdots, \tilde{A}_n^{(r-1)}).$$

The sequence $\{\tilde{A}_i^{(r)}\}$ have the same limit for all $i = 1, 2, \cdots, n$, so it's expressed by

$$A[n,t](A_1, A_2, \cdots, A_n) = \lim_{r\to\infty} \tilde{A}_i^{(r)}.$$

Here we introduce the following power means: for positive invertible operators $A_1, A_2, \cdots, A_n$, define

$$F(r) = \begin{cases} (\nabla_\alpha(A_1^r, A_2^r, \cdots, A_n^r))^{\frac{1}{r}}, & r \neq 0; \\ \exp(\nabla_\alpha(\log A_1, \log A_2, \cdots, \log A_n)), & r = 0. \end{cases}$$

It is clear that $F(r)$ is monotone increasing under the chaotic order, but is not monotone under the usual order. Besides, $F(0)$ is called chaotically geometric mean for $A_1, A_2, \cdots, A_n$.

Theorem A [6–8] Let $0 < m \leq A_i \leq M$ with $m < M$ for $i = 1, 2, \cdots, n$ and $\sum\limits_{i=1}^{n} \| x_i \|^2 = 1$. If $f(t)$ is a positive real valued continuous convex function on $[m, M]$, then

$$f(\sum_{i=1}^{n}(A_i x_i, x_i)) \leq \sum_{i=1}^{n}(f(A_i)x_i, x_i) \leq \lambda(m, M, f) f(\sum_{i=1}^{n}(A_i x_i, x_i)), \tag{1.2}$$

where

$$\lambda(m, M, f) = \max\{\frac{1}{f(t)}[\frac{f(M) - f(m)}{M - m}(t - m) + f(m)]; \quad t \in [m, M]\}. \tag{1.3}$$

Theorem B [4] Let A and B be positive operators satisfying $0 < m \leq A \leq M$ for some scalars $m < M$. If $0 \leq A \leq B$, then

$$A^p \leq K_+(h,p) B^p, \quad p \geq 1,$$

where

$$K_+(h,p) = \frac{(p-1)^{p-1}}{p^p} \frac{(h^p - 1)^p}{(h-1)(h^p-1)^{p-1}}, \quad h = \frac{M}{m}. \tag{1.4}$$

Theorem C [3] If $0 < m \leq A_i \leq M$ with $m < M$, $i = 2, \cdots, n$ $(n \geq 2)$, then

$$G[n,t](A_1, A_2, \cdots, A_n) \leq A[n,t](A_1, A_2, \cdots, A_n) \leq K(h,2) G[n,t](A_1, A_2, \cdots, A_n),$$

where

$$K(h,2) = \frac{(h+1)^2}{4h}, \quad h = \frac{M}{m}.$$

2 Boundedness of the Operator Convexity for Convex Functions

Based on the previous results coming from the Mond-Pečarič method we obtain the ratio type inequalities as follows.

Theorem 2.1 Let $0 < m \leq A_i \leq M$ with $m < M$ for $i = 1, 2, \cdots, n$. If $f(t)$ is a positive real valued continuous convex function on $[m, M]$, then

$$\begin{aligned} \frac{1}{\lambda(m,M,f)} f(\nabla_\alpha(A_1, A_2, \cdots, A_n)) &\leq \nabla_\alpha(f(A_1), f(A_2), \cdots, f(A_n)) \\ &\leq \lambda(m,M,f) f(\nabla_\alpha(A_1, A_2, \cdots, A_n)). \end{aligned}$$

Proof For unit vector $x \in H$, put $x_i = \sqrt{\alpha_i}x, i = 1, 2, \cdots, n$ in Theorem A, then

$$\sum_{i=1}^{n} \alpha_i(f(A_i)x, x) \leq \lambda(m,M,f) f(\sum_{i=1}^{n} \alpha_i(A_i x, x)).$$

Since $f(t)$ is a positive real valued continuous convex function on $[m, M]$, (1.2) leads to

$$((\sum_{i=1}^{n} \alpha_i f(A_i))x, x) \leq \lambda(m,M,f) f((\sum_{i=1}^{n} \alpha_i A_i x, x)) \leq \lambda(m,M,f)(f(\sum_{i=1}^{n} \alpha_i A_i)x, x).$$

Thus we have

$$\nabla_\alpha(f(A_1), f(A_2), \cdots, f(A_n)) \leq \lambda(m,M,f) f(\nabla_\alpha(A_1, A_2, \cdots, A_n)).$$

Next, since $0 < m \leq \sum_{i=1}^{n} \alpha_i A_i \leq M$ and $f(t)$ be a positive real valued continuous convex function on $[m, M]$, it follows from (1.2) that

$$\begin{aligned} (\nabla_\alpha(f(A_1), f(A_2), \cdots, f(A_n))x, x) &= \sum_{i=1}^{n} \alpha_i(f(A_i)x, x) \geq f(\sum_{i=1}^{n} \alpha_i(A_i x, x)) \\ &= f(\nabla_\alpha(A_1, A_2, \cdots, A_n)x, x) \\ &\geq \frac{1}{\lambda(m,M,f)}(f(\nabla_\alpha(A_1, A_2, \cdots, A_n))x, x). \end{aligned}$$

Therefore we have

$$\lambda(m,M,f)\nabla_\alpha(f(A_1), f(A_2), \cdots, f(A_n)) \geq f(\nabla_\alpha(A_1, A_2, \cdots, A_n)).$$

By the same way we can get the counterpart of Theorem 2.1.

Theorem 2.2 Let $0 < m \leq A_i \leq M$ with $m < M$ for $i = 1, 2, \cdots, n$. If $f(t)$ is a positive real valued continuous concave function on $[m, M]$, then

$$\begin{aligned} \frac{1}{\mu(m,M,f)} f(\nabla_\alpha(A_1, A_2, \cdots, A_n)) &\geq \nabla_\alpha(f(A_1), f(A_2), \cdots, f(A_n)) \\ &\geq \mu(m,M,f) f(\nabla_\alpha(A_1, A_2, \cdots, A_n)), \end{aligned}$$

where $\mu(m,M,f)=\min\{\frac{1}{f(t)}(\frac{f(M)-f(m)}{M-m}(t-m)+f(m)),\quad t\in[m,M]\}$.

3 The Main Applications

Theorem 3.1 Let $0<m\le A_i\le M$ with $m<M$ for $i=1,2,\cdots,n$.

(i) If $0<r\le 1$, then

$$(\nabla_\alpha(A_1,A_2,\cdots,A_n))^r\ge\nabla_\alpha(A_1^r,A_2^r,\cdots,A_n^r)\ge K_+(h^r,\frac{1}{r})^{-r}(\nabla_\alpha(A_1,A_2,\cdots,A_n))^r;$$

(ii) If $1\le r\le 2$, then

$$(\nabla_\alpha(A_1,A_2,\cdots,A_n))^r\le\nabla_\alpha(A_1^r,A_2^r,\cdots,A_n^r)\le K_+(h,r)(\nabla_\alpha(A_1,A_2,\cdots,A_n))^r;$$

(iii) If $r>2$, then

$$\frac{1}{K_+(h,r)}(\nabla_\alpha(A_1,A_2,\cdots,A_n))^r\le\nabla_\alpha(A_1^r,A_2^r,\cdots,A_n^r)\le K_+(h,r)(\nabla_\alpha(A_1,A_2,\cdots,A_n))^r.$$

Proof Put $f(t)=t^r$, we distinguish three cases.

In the case of $0<r\le 1$, $f(t)=t^r$ is operator concave, $\mu(m,M,f)=K_+(h^r,\frac{1}{r})^{-r}$, Theorem 2.2 and the definition of operator concavity of t^r lead to (i).

In the case of $1\le r\le 2$, $f(t)=t^r$ is operator convex, $\lambda(m,M,f)=K_+(h,r)$, following from Theorem 2.1 and the definition of operator convex, we have (ii).

In the case of $r>2$, $f(t)=t^r$ is not operator convex, but is convex, so Theorem 2.1 yields (iii).

Theorem 3.2 Let $0<m\le A_i\le M$ with $m<M$ for $i=1,2,\cdots,n$, and $0<r\le s$.

(i) If $0<r\le 1$, then

$$K_+(h^r,\frac{1}{r})^{-1}K_+(h^r,\frac{s}{r})^{-\frac{1}{s}}F(s)\le F(r)\le K_+(h^r,\frac{1}{r})F(s);$$

(ii) If $r\ge 1$, then

$$K_+(h^r,\frac{s}{r})^{-\frac{1}{s}}F(s)\le F(r)\le F(s).$$

Proof Since $0<\frac{r}{s}\le 1$, $m^s\le A_i^s\le M^s$, it follows from Theorem 3.1 that

$$\begin{aligned}(\nabla_\alpha(A_1^s,A_2^s,\cdots,A_n^s))^{\frac{r}{s}} &\ge (\nabla_\alpha(A_1^r,A_2^r,\cdots,A_n^r))\\ &\ge K_+(h^r,\frac{s}{r})^{-\frac{r}{s}}(\nabla_\alpha(A_1^s,A_2^s,\cdots,A_n^s))^{\frac{r}{s}}.\end{aligned}\tag{3.1}$$

If $r\ge 1$, then $0<\frac{1}{r}\le 1$, by raising all terms of (3.1) to $\frac{1}{r}$ it follows from operator monotonity of $x^{\frac{1}{r}}$ that

$$(\nabla_\alpha(A_1^s,A_2^s,\cdots,A_n^s))^{\frac{1}{s}}\ge(\nabla_\alpha(A_1^r,A_2^r,\cdots,A_n^r))^{\frac{1}{r}}\ge K_+(h^r,\frac{s}{r})^{-\frac{1}{s}}(\nabla_\alpha(A_1^s,A_2^s,\cdots,A_n^s))^{\frac{1}{s}}.$$

If $0<r\le 1$, then $\frac{1}{r}\ge 1$, Theorem B and (3.1) lead to

$$\begin{aligned}K_+(h^r,\frac{1}{r})(\nabla_\alpha(A_1^s,A_2^s,\cdots,A_n^s))^{\frac{1}{s}} &\ge \nabla_\alpha(A_1^r,A_2^r,\cdots,A_n^r)^{\frac{1}{r}}\\ &\ge K_+(h^r,\frac{1}{r})^{-1}K_+(h^r,\frac{s}{r})^{-\frac{1}{s}}(\nabla_\alpha(A_1^s,A_2^s,\cdots,A_n^s))^{\frac{1}{s}}.\end{aligned}$$

Theorem 3.3 If $0 < m \le A_i \le M$ with $m < M$ for $i = 1, 2, \cdots, n$, then

$$\frac{1}{M_h(1)} e^{\nabla_\alpha(A_1, A_2, \cdots, A_n)} \le \nabla_\alpha(e^{A_1}, e^{A_2}, \cdots, e^{A_n}) \le M_h(1) e^{\nabla_\alpha(A_1, A_2, \cdots, A_n)},$$

where $M_h(1) = \frac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}}$, $h = e^{M-m}$.

Proof Put $f(t) = e^t$ in Theorem 2.1, then the required inequalities hold since

$$\begin{aligned} \lambda(m, M, e^t) &= \frac{1}{e^{t_0}}\left(\frac{e^M - e^m}{M - m}(t_0 - m) + e^m\right), \quad \text{where} \quad t_0 = \frac{(m+1)e^M - (M+1)e^m}{e^M - e^m} \\ &= \frac{e^M - e^m}{M - m} e^{-\frac{(m+1)e^M - (M+1)e^m}{e^M - e^m}} = \frac{h-1}{\log h} e^{-1 + \frac{M-m}{h-1}} = \frac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}} = M_h(1). \end{aligned}$$

Theorem 3.4 If $0 < m \le A_i \le M$ with $m < M$ for $i = 1, 2, \cdots, n$, then

$$\frac{1}{M_h(1)} \nabla_\alpha(A_1, A_2, \cdots, A_n) \le \Diamond_\alpha(A_1, A_2, \cdots, A_n) \le M_h(1) \nabla_\alpha(A_1, A_2, \cdots, A_n),$$

where $h = \frac{M}{m}$.

Proof Replace A_i by $\log A_i$ in Theorem 3.3, then $h = \exp(\log M - \log m) = \frac{M}{m}$, and

$$\begin{aligned} &\frac{1}{M_h(1)} \exp\left(\sum_{i=1}^n \alpha_i \log A_i\right) \le \sum_{i=1}^n \alpha_i A_i \le M_h(1) \exp\left(\sum_{i=1}^n \alpha_i \log A_i\right), \\ &\frac{1}{M_h(1)} \Diamond_\alpha(A_1, A_2, \cdots, A_n) \le \nabla_\alpha(A_1, A_2, \cdots, A_n) \le M_h(1) \Diamond_\alpha(A_1, A_2, \cdots, A_n). \end{aligned}$$

Theorem 3.5 Let $0 < m \le A_i \le M$ with $m < M$ for $i = 1, 2, \cdots, n$, and $0 \le t \le 1$.

(i) If $0 < r \le 1$, then

$$\begin{aligned} K_+(h^r, \frac{1}{r})^{-r} K(h^r, 2)^{-1} (G_{[n,t]}(A_1, \cdots, A_n))^r &\le G_{[n,t]}(A_1^r, \cdots, A_n^r) \\ &\le K(h, 2)^r (G_{[n,t]}(A_1, \cdots, A_n))^r. \end{aligned}$$

(ii) If $1 \le r \le 2$, then

$$\begin{aligned} K_+(h, r)^{-1} K(h^r, 2)^{-1} (G_{[n,t]}(A_1, \cdots, A_n))^r &\le G_{[n,t]}(A_1^r, \cdots, A_n^r) \\ &\le K_+(h, r)^2 K(h, 2)^r (G_{[n,t]}(A_1, \cdots, A_n))^r. \end{aligned}$$

(iii) If $r > 2$, then

$$\begin{aligned} K_+(h, r)^{-2} K(h^r, 2)^{-1} (G_{[n,t]}(A_1, \cdots, A_n))^r &\le G_{[n,t]}(A_1^r, \cdots, A_n^r) \\ &\le K_+(h, r)^2 K(h, 2)^r (G_{[n,t]}(A_1, \cdots, A_n))^r. \end{aligned}$$

Proof If $0 < r \le 1$, then Theorem C implies that

$$K(h, 2) G_{[n,t]}(A_1, \cdots, A_n) \ge A_{[n,t]}(A_1, \cdots, A_n) \ge G_{[n,t]}(A_1, \cdots, A_n). \tag{3.2}$$

By raising all terms of (3.2) to the power r, and notice that t^r is operator concave for $0 < r \leq 1$, then we have

$$\begin{aligned} K(h,2)^r(G_{[n,t]}(A_1,\cdots,A_n))^r &\geq (A_{[n,t]}(A_1,\cdots,A_n))^r \\ &\geq A_{[n,t]}(A_1^r,\cdots,A_n^r) \\ &\geq G_{[n,t]}(A_1^r,\cdots,A_n^r). \end{aligned}$$

Next, replace A_i by A_i^r in (3.2) for all $i = 0, 1, \cdots, n$, it follows from Theorem 3.1 that

$$\begin{aligned} K(h^r,2)G_{[n,t]}(A_1^r,\cdots,A_n^r) &\geq A_{[n,t]}(A_1^r,\cdots,A_n^r) \\ &\geq K_+(h^r,\frac{1}{r})^{-r}A_{[n,t]}(A_1,\cdots,A_n)^r \\ &\geq K_+(h^r,\frac{1}{r})^{-r}(G_{[n,t]}(A_1,\cdots,A_n))^r. \end{aligned}$$

Therefore, we have

$$K_+(h^r,\frac{1}{r})^{-r}K(h^r,2)^{-1}(G_{[n,t]}(A_1,\cdots,A_n))^r \leq G_{[n,t]}(A_1^r,\cdots,A_n^r).$$

If $r \geq 1$, then, it follows from (3.2) and Theorem B that

$$K_+(h,r)K(h,2)^r(G_{[n,t]}(A_1,\cdots,A_n))^r \geq (A_{[n,t]}(A_1,\cdots,A_n))^r, \tag{3.3}$$

which leads to the following inequalities by (ii) of Theorem 3.1 that

$$(A_{[n,t]}(A_1,\cdots,A_n))^r \geq \frac{1}{K_+(h,r)}(A_{[n,t]}(A_1^r,\cdots,A_n^r)) \geq \frac{1}{K_+(h,r)}(G_{[n,t]}(A_1^r,\cdots,A_n^r)). \tag{3.4}$$

(ii) Replace A_i^r by A_i in (3.3), since t^r is operator convex for $1 < r \leq 2$, we have

$$\begin{aligned} K(h^r,2)G_{[n,t]}(A_1^r,\cdots,A_n^r) &\geq A_{[n,t]}(A_1^r,\cdots,A_n^r) \\ &\geq A_{[n,t]}(A_1,\cdots,A_n)^r \\ &\geq \frac{1}{K_+(h,r)}G_{[n,t]}(A_1,\cdots,A_n)^r. \end{aligned}$$

On the other hand, we can get the second inequality from (3.4) and (3.5) immediately.

(iii) If $r > 2$, it follows from (3.3), (3.4) and (3.5) that

$$\begin{aligned} K(h^r,2)G_{[n,t]}(A_1^r,\cdots,A_n^r) &\geq A_{[n,t]}(A_1^r,\cdots,A_n^r) \\ &\geq \frac{1}{K_+(h,r)}A_{[n,t]}(A_1,\cdots,A_n)^r \\ &\geq \frac{1}{K_+(h,r)^2}G_{[n,t]}(A_1,\cdots,A_n)^r. \end{aligned}$$

The second inequality follows from (3.4) and (3.5) immediately.

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凸函数的算子凸性估计及其在算子平均中的应用

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摘 要: 本文利用Mond-Pečarić方法对凸函数的n个算子凸性进行了比式估计. 在此基础上, 得到了n个代数平均的幂与n个算子的幂的代数平均的比较, n个算子的几何平均的幂与n个算子的幂的几何平均的比较. 特别地, 文中还给出了代数平均和混序几何平均.

关键词: 算子凸; 代数平均; 几何平均; 混序几何平均

MR(2010)**主题分类号:** 47A56; 47L25 **中图分类号:** O177.1