QUASI-HADAMARD PRODUCT OF MEROMORPHIC UNIVALENT FUNCTIONS AT INFINITY

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Abstract: In this paper, we study quasi-Hadamard product problem for certain new sub-classes of meromorphic starlike and convex functions in the punctured disk $U^*$. By using the method of convolution, we derive some results associated with the quasi-Hadamard product of functions belonging to these subclasses, which generalizes some known results.

Keywords: analytic functions; meromorphic; starlike; convex; quasi-Hadamard product

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1 Introduction

Let $\Sigma$ denote the class of functions $f$ of the form
\[ f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \]  
which are analytic in the punctured disk $U^* = \{z : 0 < |z| < 1\}$.

Also let $\Sigma_\alpha$ denote the class of functions of the form
\[ F(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^{n-\frac{\alpha}{n}} \quad (\alpha \in N \setminus \{1\}), \]  
which are analytic in the punctured disk $U^*$ (cf. [1, 2]). When $\alpha$ goes to infinity then $(n - \frac{\alpha}{n})$ approaches $n$; hence $\Sigma_\alpha = \Sigma$.

Throughout this paper, let the functions of the form
\[ f(z) = \frac{a_0}{z} + \sum_{n=1}^{\infty} a_n z^{n-\frac{\alpha}{n}} \quad (a_0 > 0, a_n \geq 0, \alpha \in N \setminus \{1\}), \]  
\[ f_i(z) = \frac{a_{0,i}}{z} + \sum_{n=1}^{\infty} a_{n,i} z^{n-\frac{\alpha}{n}} \quad (a_{0,i} > 0, a_{n,i} \geq 0, \alpha \in N \setminus \{1\}), \]
\[
g(z) = \frac{b_0}{z} + \sum_{n=1}^{\infty} b_n z^{n-\frac{3}{2}} \quad (b_0 > 0, b_n \geq 0, \alpha \in N \setminus \{1\}),
\]
and
\[
g_j(z) = \frac{b_{0,j}}{z} + \sum_{n=1}^{\infty} b_{n,j} z^{n-\frac{3}{2}} \quad (b_{0,j} > 0, b_{n,j} \geq 0, \alpha \in N \setminus \{1\})
\]
be regular and univalent in the punctured disk \(U^*\).

For the function \(F \in \Sigma_\alpha\), we define
\[
I_0^\alpha F(z) = F(z),
\]
\[
I_1^\alpha F(z) = zF'(z) + \frac{2}{z},
\]
and for \(k = 1, 2, \ldots\), we can write
\[
I_k^\alpha F(z) = z(I_{k-1}^\alpha F(z))' + \frac{2}{z},
\]
where \(\alpha \in N \setminus \{1\}, k \geq 0\) and \(z \in U^*\). We note that when \(\alpha\) goes to \(\infty\) then \(n(\frac{\alpha - 1}{\alpha})\) approaches \(n\); in this way we have \(I_n^k \rightarrow I_k\), which was introduced by Frasin and Darus [3] (see also [4]).

With the help of the differential operator \(I_n^k\), we define the following subclasses of \(\Sigma_\alpha\).

Let \(\Sigma_\alpha S^*(k, \beta, \gamma)\) be the class of functions \(F\) defined by (1.2) and satisfying the condition
\[
\left| \frac{z(I_k^\alpha F(z))'}{I_k^\alpha F(z)} + 1 \right| < \beta \left| \frac{z(I_k^\alpha F(z))'}{I_k^\alpha F(z)} + 2\gamma - 1 \right|
\]
\((z \in U^*, 0 \leq \gamma < 1, 0 < \beta \leq 1, k \in \mathbb{N_0} = N \cup \{0\}, \alpha \in N \setminus \{1\}).

Also let \(\Sigma_\alpha C(k, \beta, \gamma)\) be the class of functions \(F\) for which \(-zF'(z) \in \Sigma_\alpha S^*(k, \beta, \gamma)\).

Using similar methods as given in [4], we can easily obtain the characterization properties for the classes \(\Sigma_\alpha S^*(k, \beta, \gamma)\) and \(\Sigma_\alpha C(k, \beta, \gamma)\) as follows.

**Lemma 1.1** A function \(f\) defined by (1.3) belongs to the class \(\Sigma_\alpha S^*(k, \beta, \gamma)\), if and only if
\[
\sum_{n=1}^{\infty} \left[n \left(\frac{\alpha - 1}{\alpha}\right)^k \left[n \left(\frac{\alpha - 1}{\alpha}\right)(1 + \beta) + (2\gamma - 1)\beta + 1\right] a_n \right] \leq 2\beta(1 - \gamma)a_0.
\]

**Lemma 1.2** A function \(f\) defined by (1.3) belongs to the class \(\Sigma_\alpha C(k, \beta, \gamma)\) if and only if
\[
\sum_{n=1}^{\infty} \left[n \left(\frac{\alpha - 1}{\alpha}\right)^{k+1} \left[n \left(\frac{\alpha - 1}{\alpha}\right)(1 + \beta) + (2\gamma - 1)\beta + 1\right] a_n \right] \leq 2\beta(1 - \gamma)a_0.
\]
We also note that when $\alpha$ goes to $\infty$ then we have $\Sigma_\alpha S^*(k, \beta, \gamma) \rightarrow \Sigma S^*(k, \beta, \gamma)$ and $\Sigma_\alpha C(k, \beta, \gamma) \rightarrow \Sigma C(k, \beta, \gamma)$, which are special classes that were introduced by El-Ashwah and Aouf [4].

Now, we introduce the following class of meromorphic univalent functions in $U^*$.

**Definition 1.1** A function $f$ of form (1.3), which is analytic in $U^*$, belongs to the class $\Sigma^h_\alpha(\beta, \gamma)$ if and only if

$$
\sum_{n=1}^{\infty} \left[ n \left( \frac{1 - \alpha}{\alpha} \right)^h \left( n \left( \frac{1 - \alpha}{\alpha} \right) (1 + \beta) + (2\gamma - 1)\beta + 1 \right) a_n \right] \leq 2\beta(1 - \gamma)a_0,
$$

(1.10)

where $0 \leq \gamma < 1$, $0 < \beta \leq 1$, $\alpha \in N \setminus \{1\}$ and $h$ is any fixed nonnegative real number. The class $\Sigma^h_\alpha(\beta, \gamma)$ is nonempty for any nonnegative real number $h$ as the functions have the form

$$
f(z) = \frac{a_0}{z} + \sum_{n=1}^{\infty} \frac{2\beta(1 - \gamma)a_0}{n \left( \frac{1 - \alpha}{\alpha} \right)^h \left( n \left( \frac{1 - \alpha}{\alpha} \right) (1 + \beta) + (2\gamma - 1)\beta + 1 \right)} \lambda_n z^{-\frac{n}{h}}, \quad (1.11)
$$

where $a_0 > 0$, $\lambda_n \geq 0$ and $\sum_{n=1}^{\infty} \lambda_n \leq 1$, satisfying inequality (1.10).

Clearly, we have the following relationships:

(i) $\Sigma^h_\alpha(\beta, \gamma) \equiv \Sigma_\alpha S^*(k, \beta, \gamma)$ and $\Sigma^{h+1}_\alpha(\beta, \gamma) \equiv \Sigma_\alpha C(k, \beta, \gamma)$;

(ii) $\Sigma^{h_1}_\alpha(\beta, \gamma) \subset \Sigma^{h_2}_\alpha(\beta, \gamma)$ ($h_1 > h_2 \geq 0$);

(iii) $\Sigma^0_\alpha(\beta, \gamma) \subset \Sigma^{h-1}_\alpha(\beta, \gamma) \subset \cdots \subset \Sigma_\alpha S^*(k, \beta, \gamma) \subset \Sigma_\alpha C(k, \beta, \gamma)$ ($h > k + 1$).

Following the earlier works of Mogra [5, 6] and Aouf and Darwish [7] (see also [4, 8]), we define the quasi-Hadamard product of the functions $f(z)$ and $g(z)$ by

$$
f \ast g(z) = \frac{a_0 b_0}{z} + \sum_{n=1}^{\infty} a_n b_n z^{-\frac{n}{h}} \quad (1.12)
$$

Similarly, we can define the quasi-Hadamard product of more than two functions, e.g.,

$$
f_1 \ast f_2 \ast \cdots \ast f_p(z) = \left( \prod_{i=1}^{p} a_{0,i} \right) z^{-1} + \sum_{n=1}^{\infty} \left( \prod_{i=1}^{p} a_{n,i} \right) z^{-\frac{n}{h}} \quad (1.13)
$$

where the functions $f_i$ ($i = 1, 2, \cdots, p$) are given by (1.4).

The object of this paper is to derive certain results related to the quasi-Hadamard product of functions belonging to the classes $\Sigma^h_\alpha(\beta, \gamma)$, $\Sigma_\alpha S^*(k, \beta, \gamma)$ and $\Sigma_\alpha C(k, \beta, \gamma)$.

## 2 Main Results

Unless otherwise mentioned, we shall assume throughout the following results that $z \in U^*$, $0 \leq \gamma < 1$, $0 < \beta \leq 1$, $k \in N_0$, $\alpha \in N \setminus \{1\}$ and $h$ is any fixed nonnegative real number.

**Theorem 2.1** Let the functions $f_i(z)$ defined by (1.4) be in the class $\Sigma_\alpha C(k, \beta, \gamma)$ for every $i = 1, 2, \cdots, p$; and let the functions $g_j(z)$ defined by (1.6) be in the class $\Sigma^{(k+2)+q(h+1)-1}_\alpha(\beta, \gamma)$ for every $j = 1, 2, \cdots, q$. Then the quasi-Hadamard product $f_1 \ast f_2 \ast \cdots \ast f_p \ast g_1 \ast g_2 \ast \cdots \ast g_q(z)$ belongs to the class $\Sigma^{p(k+2)+q(h+1)-1}_\alpha(\beta, \gamma)$. 

Proof Let $G(z) = f_1 * f_2 * \cdots * f_p * g_1 * g_2 * \cdots * g_q(z)$, then
\[ G(z) = \left( \prod_{i=1}^{p} a_{0,i} \prod_{j=1}^{q} b_{0,j} \right) z^{-1} + \sum_{n=1}^{\infty} \left( \prod_{i=1}^{p} a_{n,i} \prod_{j=1}^{q} b_{n,j} \right) z^{n - \frac{2}{\alpha}}. \] (2.1)

It is sufficient to show that
\[ \sum_{n=1}^{\infty} \left\{ \left[ n \left( \frac{\alpha - 1}{\alpha} \right) \right]^{p(k+2) + q(h+1) - 1} \left[ n \left( \frac{\alpha - 1}{\alpha} \right) (1 + \beta) + (2\gamma - 1)\beta + 1 \right] \left( \prod_{i=1}^{p} a_{n,i} \prod_{j=1}^{q} b_{n,j} \right) \right\} \leq 2\beta(1 - \gamma) \left( \prod_{i=1}^{p} a_{0,i} \prod_{j=1}^{q} b_{0,j} \right). \] (2.2)

Since $f_i \in \Sigma_{\alpha} C(k, \beta, \gamma)$, by Lemma 1.2 we have
\[ \sum_{n=1}^{\infty} \left[ n \left( \frac{\alpha - 1}{\alpha} \right) \right]^{k+1} \left[ n \left( \frac{\alpha - 1}{\alpha} \right) (1 + \beta) + (2\gamma - 1)\beta + 1 \right] a_{n,i} \leq 2\beta(1 - \gamma) a_{0,i} \] (2.3)
for every $i = 1, 2, \cdots, p$. Thus,
\[ \left[ n \left( \frac{\alpha - 1}{\alpha} \right) \right]^{k+1} \left[ n \left( \frac{\alpha - 1}{\alpha} \right) (1 + \beta) + (2\gamma - 1)\beta + 1 \right] a_{n,i} \leq 2\beta(1 - \gamma) a_{0,i} \]
or
\[ a_{n,i} \leq \frac{2\beta(1 - \gamma)}{\left[ n \left( \frac{\alpha - 1}{\alpha} \right) \right]^{k+1} \left[ n \left( \frac{\alpha - 1}{\alpha} \right) (1 + \beta) + (2\gamma - 1)\beta + 1 \right] a_{0,i}} \]
for every $i = 1, 2, \cdots, p$. The right-hand expression of the last inequality is not greater than $\left[ n \left( \frac{\alpha - 1}{\alpha} \right) \right]^{-(k+2)} a_{0,i}$. Therefore,
\[ a_{n,i} \leq \left[ n \left( \frac{\alpha - 1}{\alpha} \right) \right]^{-(k+2)} a_{0,i} \] (2.4)
for every $i = 1, 2, \cdots, p$. Also, since $g_j \in \Sigma_{\alpha} h(\beta, \gamma)$, we find from (1.10) that
\[ \sum_{n=1}^{\infty} \left[ n \left( \frac{\alpha - 1}{\alpha} \right) \right]^{h} \left[ n \left( \frac{\alpha - 1}{\alpha} \right) (1 + \beta) + (2\gamma - 1)\beta + 1 \right] b_{n,j} \leq 2\beta(1 - \gamma) b_{0,j}, \] (2.5)
which implies that
\[ b_{n,j} \leq \left[ n \left( \frac{\alpha - 1}{\alpha} \right) \right]^{-(h+1)} b_{0,j} \] (2.6)
for every $j = 1, 2, \cdots, q$.

Using (2.4)–(2.6) for $i = 1, 2, \cdots, p; j = q$; and $j = 1, 2, \cdots, q - 1$ respectively, we have
\[ \sum_{n=1}^{\infty} \left\{ \left[ n \left( \frac{\alpha - 1}{\alpha} \right) \right]^{p(k+2) + q(h+1) - 1} \left[ n \left( \frac{\alpha - 1}{\alpha} \right) (1 + \beta) + (2\gamma - 1)\beta + 1 \right] \left( \prod_{i=1}^{p} a_{n,i} \prod_{j=1}^{q} b_{n,j} \right) \right\} \leq \sum_{n=1}^{\infty} \left\{ \left[ n \left( \frac{\alpha - 1}{\alpha} \right) \right]^{p(k+2) + q(h+1) - 1} \left[ n \left( \frac{\alpha - 1}{\alpha} \right) \right]^{-(p+1)h} \left[ n \left( \frac{\alpha - 1}{\alpha} \right) \right]^{-(q+1)(h+1)} \right\} \cdot \left( \prod_{i=1}^{p} a_{0,i} \prod_{j=1}^{q} b_{0,j} \right) \left[ n \left( \frac{\alpha - 1}{\alpha} \right) (1 + \beta) + (2\gamma - 1)\beta + 1 \right] b_{n,q} \]
Thus, we have \( G(z) \in \Sigma^{p(k+2)+q(k+1)-1}_\alpha(\beta, \gamma) \). This completes the proof of Theorem 2.1.

Corollary 2.1 Let the functions \( f_i(z) \) defined by (1.4) and the functions \( g_j(z) \) defined by (1.6) belong to the class \( \Sigma_k C(k, \beta, \gamma) \) for every \( i = 1, 2, \cdots, p \) and \( j = 1, 2, \cdots, q \). Then the quasi-Hadamard product \( f_1 \ast f_2 \ast \cdots \ast f_p \ast g_1 \ast g_2 \ast \cdots \ast g_q(z) \) belongs to the class \( \Sigma^{p(k+2)+q(k+1)-1}_\alpha(\beta, \gamma) \).

Theorem 2.2 Let the functions \( f_i(z) \) defined by (1.4) be in the class \( \Sigma^h \beta, \gamma \) for every \( i = 1, 2, \cdots, p \); and let the functions \( g_j(z) \) defined by (1.6) be in the class \( \Sigma^q S^\ast(k, \beta, \gamma) \) for every \( j = 1, 2, \cdots, q \). Then the quasi-Hadamard product \( f_1 \ast f_2 \ast \cdots \ast f_p \ast g_1 \ast g_2 \ast \cdots \ast g_q(z) \) belongs to the class \( \Sigma^{p(k+2)+q(k+1)-1}_\alpha(\beta, \gamma) \).

Proof Suppose that \( G(z) \) be defined as (2.1). To prove the theorem, we need to show that

\[
\sum_{n=1}^{\infty} \left\{ n \left( \frac{\alpha-1}{\alpha} \right)^p \left( 1 + \beta + (2\gamma - 1)\beta + 1 \right) \left( \prod_{i=1}^{p} a_{n,i} \prod_{j=1}^{q} b_{n,j} \right) \right\} \\
\leq 2\beta(1 - \gamma) \prod_{i=1}^{p} a_{n,i} \prod_{j=1}^{q} b_{n,j}. \tag{2.7}
\]

Since \( f_i \in \Sigma^h \beta, \gamma \), from (1.10) we have

\[
\sum_{n=1}^{\infty} \left\{ n \left( \frac{\alpha-1}{\alpha} \right) \left( 1 + \beta + (2\gamma - 1)\beta + 1 \right) a_{n,i} \right\} \leq 2\beta(1 - \gamma) a_{0,i} \tag{2.8}
\]

which implies that

\[
a_{n,i} \leq \left( n \left( \frac{\alpha-1}{\alpha} \right) \right)^{-(h+1)} a_{0,i} \tag{2.9}
\]

for every \( i = 1, 2, \cdots, p \). Further, since \( g_j \in \Sigma^q S^\ast(k, \beta, \gamma) \), by Lemma 1.1 we have

\[
\sum_{n=1}^{\infty} \left\{ n \left( \frac{\alpha-1}{\alpha} \right)^q \left( 1 + \beta + (2\gamma - 1)\beta + 1 \right) b_{n,j} \right\} \leq 2\beta(1 - \gamma) b_{0,j} \tag{2.10}
\]

for every \( j = 1, 2, \cdots, q \). Whence we obtain

\[
b_{n,j} \leq \left( n \left( \frac{\alpha-1}{\alpha} \right) \right)^{-(k+1)} b_{0,j} \tag{2.10}
\]
for every $j = 1, 2, \ldots, q$.

Using (2.8)–(2.10) for $i = p; i = 1, 2, \ldots, p - 1$; and $j = 1, 2, \ldots, q$ respectively, we get

\[
\sum_{n=1}^{\infty} \left\{ \left[ n \left( \frac{\alpha - 1}{\alpha} \right) \right]^{p(h+1)+q(k+1)-1} \left[ n \left( \frac{\alpha - 1}{\alpha} \right) \right] (1 + \beta) + (2\gamma - 1)\beta + 1 \left[ \prod_{i=1}^{p} a_{n,i} \prod_{j=1}^{q} b_{n,j} \right] \right\}
\]

\[
\leq \sum_{n=1}^{\infty} \left\{ \left[ n \left( \frac{\alpha - 1}{\alpha} \right) \right]^{p(h+1)+q(k+1)-1} \cdot \left[ n \left( \frac{\alpha - 1}{\alpha} \right) \right]^{-1} \cdot \left[ n \left( \frac{\alpha - 1}{\alpha} \right) \right]^{-(q(k+1))} \cdot \left( \prod_{i=1}^{p} a_{o,i} \prod_{j=1}^{q} b_{o,j} \right) \right\}
\]

\[
\leq \left( \prod_{i=1}^{p} a_{o,i} \prod_{j=1}^{q} b_{o,j} \right) \left\{ \sum_{n=1}^{\infty} \left[ n \left( \frac{\alpha - 1}{\alpha} \right) \right]^{h} \left[ n \left( \frac{\alpha - 1}{\alpha} \right) \right] (1 + \beta) + (2\gamma - 1)\beta + 1 \right\}
\]

\[
\leq 2\beta(1 - \gamma) \left( \prod_{i=1}^{p} a_{o,i} \prod_{j=1}^{q} b_{o,j} \right)
\]

Therefore, we have $G(z) \in \Sigma_{\alpha}^{p(h+1)+q(k+1)-1}(\beta, \gamma)$. We complete the proof.

By taking $h = k$ in Theorem 2.2, we get the following result.

**Corollary 2.2** Let the functions $f_{i}(z)$ defined by (1.4) and the functions $g_{j}(z)$ defined by (1.6) belong to the class $\Sigma_{\alpha}S^{*}(k, \beta, \gamma)$ for every $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, q$. Then the quasi-Hadamard product $f_{1} * f_{2} * \cdots * f_{p} * g_{1} * g_{2} * \cdots * g_{q}(z)$ belongs to the class $\Sigma_{\alpha}^{p(k+1)+q(k+1)-1}(\beta, \gamma)$.

By putting $h = k$ in Theorem 2.1 or $h = k + 1$ in Theorem 2.2, we obtain the following result.

**Corollary 2.3** Let the functions $f_{i}(z)$ defined by (1.4) be in the class $\Sigma_{\alpha}C(k, \beta, \gamma)$ for every $i = 1, 2, \ldots, p$; and let the functions $g_{j}(z)$ defined by (1.6) be in the class $\Sigma_{\alpha}S^{*}(k, \beta, \gamma)$ for every $j = 1, 2, \ldots, q$. Then the quasi-Hadamard product $f_{1} * f_{2} * \cdots * f_{p} * g_{1} * g_{2} * \cdots * g_{q}(z)$ belongs to the class $\Sigma_{\alpha}^{p(k+2)+q(k+1)-1}(\beta, \gamma)$.

Next, we discuss some applications of Theorems 2.1 and 2.2.

Taking into account the quasi-Hadamard product of functions $f_{1}(z), f_{2}(z), \ldots, f_{p}(z)$ only, in the proof of Theorem 2.1, and using (2.3) and (2.4) for $i = p$ and $i = 1, 2, \ldots, p - 1$, respectively, we are led to

**Corollary 2.4** Let the functions $f_{i}(z)$ defined by (1.4) belong to the class $\Sigma_{\alpha}C(k, \beta, \gamma)$ for every $i = 1, 2, \ldots, p$. Then the quasi-Hadamard product $f_{1} * f_{2} * \cdots * f_{p}(z)$ belongs to the class $\Sigma_{\alpha}^{p(k+2)-1}(\beta, \gamma)$.

Also, taking into account the quasi-Hadamard product of functions $g_{1}(z), g_{2}(z), \ldots, g_{q}(z)$ only, in the proof of Theorem 2.2, and using (2.10) and (2.11) for $j = q$ and $j = 1, 2, \ldots, q - 1$, respectively, we are led to

**Corollary 2.5** Let the functions $g_{j}(z)$ defined by (1.6) belong to the class $\Sigma_{\alpha}S^{*}(k, \beta, \gamma)$ for every $j = 1, 2, \ldots, q$. Then the quasi-Hadamard product $g_{1} * g_{2} * \cdots * g_{q}(z)$ belongs to the class $\Sigma_{\alpha}^{q(k+1)-1}(\beta, \gamma)$.

**Remark 2.1** By letting $\alpha \to \infty$ in the proofs of Corollaries 3–5, we obtain the results obtained by El-Ashwah and Aouf [4, Theorems 3, 1 and 2, respectively].
Remark 2.2 By letting $\alpha \to \infty$ and $k = 0$ in the proofs of Corollaries 3–5, we obtain the results obtained by Mogra [6, Theorems 3, 1 and 2, respectively].

References


 inflict单叶函数的拟Hadamard卷积

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摘要: 本文研究了圆环域$U^*$内亚纯星象和凸象函数的某些子类的拟Hadamard卷积. 利用卷积方法, 获得了该类函数的拟Hadamard卷积有关的某些性质, 广泛了一些已知结果.
关键词: 解析函数; 亚纯; 星象; 凸象; 拟Hadamard卷积