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FINITE TIME STABILITY OF FRACTIONAL ORDER NEUTRAL DIFFERENTIAL EQUATIONS

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Abstract: A class of fractional order neutral differential equations is considered. By using the method of steps and the theory of differential inequalities, the existence and uniqueness theorems and the finite time stability theorem of the fractional order neutral differential equations are obtained, which extends some corresponding results in [8, 9].

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1 Introduction

Recently, the subject of fractional order delay differential equations is gaining much importance and attention. For details and examples, see [1–9] and the references therein.

Stability analysis is always one of the most important issues for differential equations, although this problem was investigated for time-delay differential equations over many years. Comparing with classical Lyapunov stability, finite-time stability (FTS) is a more practical concept, useful to study the behavior of the system over a finite interval of time and plays an important part in the study of the transient behavior of systems. Thus, it was widely studied in both classical differential equations and fractional order differential equations (for details and examples, see [8–13] and the references therein). However, for fractional order neutral differential equations, no much progress was seen on FTS.

In this paper, we consider fractional order neutral differential equations of the form

$$^{c}D^{\alpha}_{0+}[x(t) - Cx(t-\tau)] = Ax(t) + Bx(t-\tau) + f(t), \tag{1}$$

with associated function of initial state:

$$x(t) = \varphi(t), -\tau \le t \le 0, \tag{2}$$

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where ${}^cD_{0+}^{\alpha}$ is the Caputo fractional derivative of order $\alpha(0<\alpha\le 1), A, B, C\in\mathbb{R}^{n\times n}, f\in C(\mathbb{R},\mathbb{R}^n)$ and $\varphi\in C^1([-\tau,0],\mathbb{R}^n)$. We study the FTS of such differential equations. In details, we briefly introduce the definitions and properties of the fractional derivative and the fractional integral in Section 2. In Section 3, the existence and uniqueness theorems and FTS theorem are proved.

2 Preliminaries

Definitions of fractional order detivative/integral and their properties (see [14–16]) were given below.

Definition 2.1 The fractional order integral of the function $f\in L^1([a,b],\mathbb{R})$ of order $\alpha\in\mathbb{R}^+$ is defined by

$$I_{a+}^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}f(s)ds,$$

where $\Gamma(\cdot)$ is the gamma function, and we have

$$I_{a+}^{\alpha}((t-a)^s)=\frac{\Gamma(s+1)}{\Gamma(s+\alpha+1)}(t-a)^{s+\alpha},\quad s>-1.$$

Definition 2.2 For a function f given on the interval $[a,b]$, αth Riemann-Liouville fractional order derivative of f, is defined by

$$D_{a+}^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}(\frac{d}{dt})^n\int_a^t(t-s)^{n-\alpha-1}f(s)ds,$$

where $n=[\alpha]+1$ and $[\alpha]$ denotes the integer part of α.

Definition 2.3 For a function f given on the interval $[a,b]$, αth Caputo fractional order derivative of f, is defined by

$${}^cD_{a+}^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t(t-s)^{n-\alpha-1}f^{(n)}(s)ds.$$

Here $n=[\alpha]+1$.

In further discussion we will denote $I_{0+}^{\alpha}f(t)$ and ${}^cD_{0+}^{\alpha}f(t)$ as $I^{\alpha}f(t)$ and $D^{\alpha}f(t)$, respectively. Note that (see [15])

(1) $I^{\alpha}I^{\beta}f(t)=I^{\alpha+\beta}f(t),\quad \alpha,\beta\ge 0.$

(2) $I^{\alpha}t^s=\frac{\Gamma(s+1)}{\Gamma(s+\alpha+1)}t^{s+\alpha},\quad \alpha>0, s>-1, t>0.$

(3) $D^{\alpha}(I^{\alpha}f(t))=f(t),\quad n-1<\alpha\le n, n\in\mathbb{N}.$

(4) $I^{\alpha}(D^{\alpha}f(t))=f(t)-\sum\limits_{k=0}^{n-1}f^{(k)}(0^+)\frac{t^k}{k!},\quad n-1<\alpha\le n, n\in\mathbb{N}.$

The following lemmas play major role in our analysis.

Lemma 2.4 Let $u\ge v\ge 0$.

(1) If $r\ge 1$, then $(u-v)^r\le u^r-v^r$.

(2) If $0<r<1$, then $(u-v)^r\ge u^r-v^r$.

Proof (1) Let $r \geq 1$, then

$$\begin{aligned}\int_0^u (u-s)^{r-1}ds &= \int_0^{u-v}(u-s)^{r-1}ds + \int_{u-v}^u (u-s)^{r-1}ds \\ &\geq \int_0^{u-v}(u-v-s)^{r-1}ds + \int_{u-v}^u (u-s)^{r-1}ds.\end{aligned}$$

That is $(u-v)^r \leq u^r - v^r, u \geq v \geq 0$.

(2) Let $0 < r < 1$, then

$$\begin{aligned}\int_0^{u-v}(u-v-s)^{r-1}ds &\geq \int_0^{u-v}(u-s)^{r-1}ds \\ &= \int_0^u (u-s)^{r-1}ds - \int_{u-v}^u (u-s)^{r-1}ds.\end{aligned}$$

That is $(u-v)^r \geq u^r - v^r, u \geq v \geq 0$. The lemma is proved.

Lemma 2.5 Let $b > c > 0, \tau \geq 0$ and $0 < \beta < 1$. Then $u(t) = bt^\beta - b(t-\tau)^\beta - c\tau^\beta$ is decreasing on $[\tau, +\infty)$ and $u(t) \in (-c\tau^\beta, (b-c)\tau^\beta]$.

Proof For $t \geq \tau$, we have

$$u'(t) = \frac{b\beta}{t^{1-\beta}} - \frac{b\beta}{(t-\tau)^{1-\beta}} \leq 0$$

and

$$\lim_{t\to+\infty} u(t) = \lim_{t\to+\infty} \frac{b - b(1-\frac{\tau}{t})^\beta - c(\frac{\tau}{t})^\beta}{(\frac{1}{t})^\beta}. \tag{3}$$

Let $s = \frac{1}{t}$, from (3), we have

$$\begin{aligned}\lim_{t\to+\infty} u(t) &= \lim_{s\to0} \tfrac{b-b(1-s\tau)^\beta - c(s\tau)^\beta}{s^\beta} \\ &= \lim_{s\to0} \frac{\frac{b\tau}{(1-s\tau)^{1-\beta}} - \frac{c\tau}{(s\tau)^{1-\beta}}}{s^{\beta-1}} \\ &= \lim_{s\to0} \frac{\tau^\beta[b(s\tau)^{1-\beta} - c(1-s\tau)^{1-\beta}]}{(1-s\tau)^{1-\beta}} \\ &= -c\tau^\beta.\end{aligned}$$

This proves the lemma.

3 Main Results

First, we consider the initial value problem (1), (2). By the method of steps, We obtain existence and uniqueness theorems for the initial value problem (1), (2).

Theorem 3.1 $x(t)$ is a continuous solution of the initial value problem (1), (2) on $[-\tau, T]$ if and only if $x(t)$ satisfies the relation

$$\begin{cases} x(t) = & \sum\limits_{i=0}^{\infty} A^i B I^{(i+1)\alpha} x(t-\tau) + \sum\limits_{i=0}^{\infty} A^i I^{(i+1)\alpha} f(t) + \sum\limits_{i=0}^{\infty} A^i C I^{i\alpha} x(t-\tau) \\ & + \sum\limits_{i=0}^{\infty} A^i I^{i\alpha}[x(0) - Cx(-\tau)], \quad t \in [0,T], \\ x(t) = & \varphi(t), \quad -\tau \leq t \leq 0, \end{cases} \tag{4}$$

where $T > 0$.

Proof Let $x(t) \in C([-\tau, T], \mathbb{R}^n)$ be the solution of the initial value problem (1), (2). Then, for $t \geq 0$, we have

$$\begin{aligned} x(t) &= AI^\alpha x(t) + BI^\alpha x(t-\tau) + I^\alpha f(t) + Cx(t-\tau) + x(0) - Cx(-\tau) \\ &= AI^\alpha [AI^\alpha x(t) + BI^\alpha x(t-\tau) + I^\alpha f(t) + Cx(t-\tau) + x(0) - Cx(-\tau)] \\ &\quad + BI^\alpha x(t-\tau) + I^\alpha f(t) + Cx(t-\tau) + x(0) - Cx(-\tau) \\ &= A^2 I^{2\alpha} x(t) + ABI^{2\alpha} x(t-\tau) + AI^{2\alpha} f(t) + ACI^\alpha x(t-\tau) + AI^\alpha [x(0) - Cx(-\tau)] \\ &\quad + BI^\alpha x(t-\tau) + I^\alpha f(t) + Cx(t-\tau) + x(0) - Cx(-\tau) \\ &= A^2 I^{2\alpha} [AI^\alpha x(t) + BI^\alpha x(t-\tau) + I^\alpha f(t) + Cx(t-\tau) + x(0) - Cx(-\tau)] \\ &\quad + ABI^{2\alpha} x(t-\tau) + AI^{2\alpha} f(t) + ACI^\alpha x(t-\tau) + AI^\alpha [x(0) - Cx(-\tau)] \\ &\quad + BI^\alpha x(t-\tau) + I^\alpha f(t) + Cx(t-\tau) + x(0) - Cx(-\tau) \\ &= A^3 I^{3\alpha} x(t) + A^2 BI^{3\alpha} x(t-\tau) + A^2 I^{3\alpha} f(t) + A^2 CI^{2\alpha} x(t-\tau) + A^2 I^{2\alpha} [x(0) - Cx(-\tau)] \\ &\quad + ABI^{2\alpha} x(t-\tau) + AI^{2\alpha} f(t) + ACI^\alpha x(t-\tau) + AI^\alpha [x(0) - Cx(-\tau)] \\ &\quad + BI^\alpha x(t-\tau) + I^\alpha f(t) + Cx(t-\tau) + x(0) - Cx(-\tau) \\ &= \cdots \\ &= A^n I^{n\alpha} x(t) + \sum_{i=0}^{n-1} A^i BI^{(i+1)\alpha} x(t-\tau) + \sum_{i=0}^{n-1} A^i I^{(i+1)\alpha} f(t) + \sum_{i=0}^{n-1} A^i CI^{i\alpha} x(t-\tau) \\ &\quad + \sum_{i=0}^{n-1} A^i I^{i\alpha} [x(0) - Cx(-\tau)]. \end{aligned} \tag{5}$$

Taking $n \to \infty$ in (5), we have$\|A^n I^{n\alpha} x(t)\| \leq \|A\| I^{n\alpha} \|x(t)\| \to 0$. Furthermore, we have

$$\begin{aligned} &\|\sum_{i=0}^{n-1} A^i BI^{(i+1)\alpha} x(t-\tau)\| \leq \|B\|\|x\| \sum_{i=0}^{n-1} \|A\|^i \tfrac{t^{(i+1)\alpha}}{\Gamma((i+1)\alpha+1)} \leq \tfrac{\|B\|\|x\|}{\|A\|} E_\alpha(\|A\| t^\alpha), \\ &\|\sum_{i=0}^{n-1} A^i I^{(i+1)\alpha} f(t)\| \leq \|f\| \sum_{i=0}^{n-1} \|A\|^i \tfrac{t^{(i+1)\alpha}}{\Gamma((i+1)\alpha+1)} \leq \tfrac{\|f\|}{\|A\|} E_\alpha(\|A\| t^\alpha), \\ &\|\sum_{i=0}^{n-1} A^i CI^{i\alpha} x(t-\tau)\| \leq \|C\|\|x\| \sum_{i=0}^{n-1} \|A\|^i \tfrac{t^{i\alpha}}{\Gamma(i\alpha+1)} \leq \|C\|\|x\| E_\alpha(\|A\| t^\alpha) \end{aligned}$$

and

$$\|\sum_{i=0}^{n-1} A^i I^{i\alpha} [x(0) - Cx(-\tau)]\| \leq (1 + \|C\|)\|\varphi\| \sum_{i=0}^{n-1} \|A\|^i \tfrac{t^{i\alpha}}{\Gamma(i\alpha+1)} \leq (1 + \|C\|)\|\varphi\| E_\alpha(\|A\| t^\alpha),$$

where $\|f\| = \max\limits_{t \in [0,T]} \|f(t)\|$, $\|f(t)\|$ be any vector norm (e.g., $= 1, 2, \infty$), $\|A\|$ denotes the induced norm of a matrix A and $E_\alpha(t) = \sum\limits_{i=0}^{\infty} \frac{t^i}{\Gamma(i\alpha+1)}$ is the Mittag-Leffler functions. Therefore

$$\begin{aligned} x(t) = &\sum_{i=0}^{\infty} A^i BI^{(i+1)\alpha} x(t-\tau) + \sum_{i=0}^{\infty} A^i I^{(i+1)\alpha} f(t) + \sum_{i=0}^{\infty} A^i CI^{i\alpha} x(t-\tau) \\ &+ \sum_{i=0}^{\infty} A^i I^{i\alpha} [x(0) - Cx(-\tau)]. \end{aligned}$$

Conversely, from the first equation of (4), we have

$$x(t)-Cx(t-\tau)=\sum_{i=0}^{\infty}A^iBI^{(i+1)\alpha}x(t-\tau)+\sum_{i=0}^{\infty}A^iI^{(i+1)\alpha}f(t)+\sum_{i=1}^{\infty}A^iCI^{i\alpha}x(t-\tau)+\sum_{i=0}^{\infty}A^iI^{i\alpha}[x(0)-Cx(-\tau)]. \tag{6}$$

Letting the operator D^α act on both sides of (6), we have

$$\begin{aligned}
&D^\alpha[x(t)-Cx(t-\tau)]=\sum_{i=0}^{\infty}A^iBI^{i\alpha}x(t-\tau)+\sum_{i=0}^{\infty}A^iI^{i\alpha}f(t)+\sum_{i=1}^{\infty}A^iCI^{(i-1)\alpha}x(t-\tau)\\
&+\sum_{i=1}^{\infty}A^iI^{(i-1)\alpha}[x(0)-Cx(-\tau)]\\
=\ &Bx(t-\tau)+f(t)+\sum_{i=1}^{\infty}A^iBI^{i\alpha}x(t-\tau)+\sum_{i=1}^{\infty}A^iI^{i\alpha}f(t)+\sum_{i=1}^{\infty}A^iCI^{(i-1)\alpha}x(t-\tau)\\
&+\sum_{i=1}^{\infty}A^iI^{(i-1)\alpha}[x(0)-Cx(-\tau)]\\
=\ &Ax(t)+Bx(t-\tau)+f(t).
\end{aligned}$$

This proves the theorem.

Next, by the method of steps, we prove existence and uniqueness theorems for the initial value problem (1), (2).

Theorem 3.2 For a given real number $T>0$, the initial value problem (1), (2) exists a unique continuous solution $x(t)$ defined on $[0,T]$ which coincides with φ on $[-\tau,0]$.

Proof From Theorem 3.1, we know the initial value problem (1), (2) is equivalent to (4). Next, we only need to prove (4) exists a unique continuous solution.

(1) For $t\in[0,\tau]$, we have

$$\begin{aligned}
x_1(t)&=\sum_{i=0}^{\infty}A^iBI^{(i+1)\alpha}x(t-\tau)+\sum_{i=0}^{\infty}A^iI^{(i+1)\alpha}f(t)+\sum_{i=0}^{\infty}A^iCI^{i\alpha}x(t-\tau)\\
&\quad+\sum_{i=0}^{\infty}A^iI^{i\alpha}[x(0)-Cx(-\tau)]\\
&=\sum_{i=0}^{\infty}A^iB\tfrac{1}{\Gamma((i+1)\alpha)}\int_0^t(t-s)^{(i+1)\alpha-1}\varphi(s-\tau)ds+\sum_{i=0}^{\infty}A^iI^{(i+1)\alpha}f(t)\\
&\quad+\sum_{i=0}^{\infty}A^iC\tfrac{1}{\Gamma(i\alpha)}\int_0^t(t-s)^{i\alpha-1}\varphi(s-\tau)ds+\sum_{i=0}^{\infty}A^iI^{i\alpha}[\varphi(0)-C\varphi(-\tau)].
\end{aligned}$$

(2) For $t\in[\tau,2\tau]$, we have

$$\begin{aligned}
x_2(t)=\ &\sum_{i=0}^{\infty}A^iI^{(i+1)\alpha}f(t)+\sum_{i=0}^{\infty}A^iI^{i\alpha}[\varphi(0)-C\varphi(-\tau)]\\
&+\sum_{i=0}^{\infty}A^iB\tfrac{1}{\Gamma((i+1)\alpha)}\int_0^\tau(t-s)^{(i+1)\alpha-1}\varphi(s-\tau)ds\\
&+\sum_{i=0}^{\infty}A^iB\tfrac{1}{\Gamma((i+1)\alpha)}\int_\tau^t(t-s)^{(i+1)\alpha-1}x_1(s-\tau)ds\\
&+\sum_{i=0}^{\infty}A^iC\tfrac{1}{\Gamma(i\alpha)}\int_0^\tau(t-s)^{i\alpha-1}\varphi(s-\tau)ds\\
&+\sum_{i=0}^{\infty}A^iC\tfrac{1}{\Gamma(i\alpha)}\int_\tau^t(t-s)^{i\alpha-1}x_1(s-\tau)ds.
\end{aligned}$$

By induction, for $t \in [(n-1)\tau, n\tau]$, we have

$$\begin{aligned} x_n(t) = \;& \sum_{i=0}^{\infty} A^i I^{(i+1)\alpha} f(t) + \sum_{i=0}^{\infty} A^i I^{i\alpha}[\varphi(0) - C\varphi(-\tau)] \\ & + \sum_{i=0}^{\infty} A^i B \tfrac{1}{\Gamma((i+1)\alpha)} \int_0^{\tau} (t-s)^{(i+1)\alpha-1} \varphi(s-\tau) ds \\ & + \sum_{i=0}^{\infty} A^i B \tfrac{1}{\Gamma((i+1)\alpha)} \int_{\tau}^{2\tau} (t-s)^{(i+1)\alpha-1} x_1(s-\tau) ds \\ & + \cdots \\ & + \sum_{i=0}^{\infty} A^i B \tfrac{1}{\Gamma((i+1)\alpha)} \int_{(n-1)\tau}^{t} (t-s)^{(i+1)\alpha-1} x_{n-1}(s-\tau) ds \\ & + \sum_{i=0}^{\infty} A^i C \tfrac{1}{\Gamma(i\alpha)} \int_0^{\tau} (t-s)^{i\alpha-1} \varphi(s-\tau) ds \\ & + \sum_{i=0}^{\infty} A^i C \tfrac{1}{\Gamma(i\alpha)} \int_{\tau}^{2\tau} (t-s)^{i\alpha-1} x_1(s-\tau) ds \\ & + \cdots \\ & + \sum_{i=0}^{\infty} A^i C \tfrac{1}{\Gamma(i\alpha)} \int_{(n-1)\tau}^{t} (t-s)^{i\alpha-1} x_{n-1}(s-\tau) ds. \end{aligned}$$

By the method of steps, we obtain (4) exists a unique continuous solution. That is the initial value problem (1), (2) exists a unique continuous solution $x(t)$ defined on $[0, T]$ which coincides with φ on $[-\tau, 0]$.

Here, we shall consider the finite time stability of systems (1), (2).

Definition 3.3 Systems (1), (2) is finite stable w.r.t $\{t_0, J, \delta, \varepsilon, M\}$ if and only if

$$\|\varphi\| < \delta$$

and

$$\|f(t)\| < M, \quad \forall t \in J = [t_0, t_0 + T]$$

implies

$$\|x(t)\| < \varepsilon, \quad \forall t \in J,$$

where $\|\varphi\| = \max\limits_{t\in[-\tau,0]} \|\varphi(t)\|$, $\delta, M, T, \varepsilon$ are positive real numbers and $\delta < \varepsilon$.

Theorem 3.4 If there exists a positive constant b_1 such that the following conditions are satisfied:

(1) $b_1 > \|A\| + \|B\|$;

(2) $(1 + 2\|C\|) e^{\frac{b_1[(T-\tau)^\alpha - T^\alpha] + (\|A\|+\|B\|)\tau^\alpha}{\Gamma(\alpha+1)}} \leq 1$;

(3) $(1 + 2\|C\|)(t-\tau)^\alpha e^{\frac{b_1(t-\tau)^\alpha + (\|A\|+\|B\|)\tau^\alpha}{\Gamma(\alpha+1)}} + \tau^\alpha e^{\frac{(\|A\|+\|B\|)\tau^\alpha}{\Gamma(\alpha+1)}} \leq t^\alpha e^{\frac{b_1 t^\alpha}{\Gamma(\alpha+1)}}$, $\forall t \in [\tau, T]$;

(4) $[1 + 2\|C\| + \frac{MT^\alpha}{\delta\Gamma(\alpha+1)}] e^{\frac{b_1 T^\alpha}{\Gamma(\alpha+1)}} \leq \frac{\varepsilon}{\delta}$,

then systems (1), (2) is finite time stable w.r.t. $\{0, J, \delta, \varepsilon, M\}$.

Proof According to the properties of the fractional calculus, we have

$$\begin{aligned} x(t) = \;& \varphi(0) + \tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [Ax(s) + Bx(s-\tau) + f(s)] ds \\ & + Cx(t-\tau) - C\varphi(-\tau), \quad t \geq 0. \end{aligned}$$

Therefore, for $t \geq 0$,

$$\begin{aligned}\|x(t)\| \leq & \frac{1}{\Gamma(\alpha)} \int_0^t |t-s|^{\alpha-1}[\|A\|\|x(s)\| + \|B\|\|x(s-\tau)\| + \|f(s)\|]ds \\ & + \|\varphi\| + \|C\|\|\varphi\| + |C\|\|x(t-\tau)\|.\end{aligned}$$

If $y(t) = \sup\limits_{-\tau \leq \theta \leq 0} |x(t+\theta)|$ and $0 \leq t \leq \tau$, then

$$\begin{aligned}\|x(t)\| \leq & (1+2\|C\|)\|\varphi\| + \frac{\|f\|t^\alpha}{\Gamma(\alpha+1)} + \frac{\|A\|}{\Gamma(\alpha)} \int_0^t |t-s|^{\alpha-1}\|y(s)\|ds \\ & + \frac{\|B\|}{\Gamma(\alpha)} \int_0^t |t-s|^{\alpha-1}\|y(s)\|ds.\end{aligned}$$

Therefore, for $0 \leq t \leq \tau$,

$$\begin{aligned}\|y(t)\| \leq & (1+2\|C\|)\|\varphi\| + \frac{\|f\|t^\alpha}{\Gamma(\alpha+1)} + \frac{\|A\|}{\Gamma(\alpha)} \int_0^t |t-s|^{\alpha-1}\|y(s)\|ds \\ & + \frac{\|B\|}{\Gamma(\alpha)} \int_0^t |t-s|^{\alpha-1}\|y(s)\|ds.\end{aligned}$$

Applying Gronwall inequality, it is easy to get

$$\|y(t)\| \leq [(1+2\|C\|)\|\varphi\| + \frac{\|f\|t^\alpha}{\Gamma(\alpha+1)}]e^{\frac{(\|A\|+\|B\|)t^\alpha}{\Gamma(\alpha+1)}}, \quad 0 \leq t \leq \tau.$$

Also, the same argument implies the following estimate

$$\|y(t)\| \leq [(1+2\|C\|)y(t_0) + \frac{\|f\|(t-t_0)^\alpha}{\Gamma(\alpha+1)}]e^{\frac{(\|A\|+\|B\|)(t-t_0)^\alpha}{\Gamma(\alpha+1)}}, \quad 0 \leq t_0 \leq t \leq t_0+\tau.$$

Next, we need to prove that

$$\|y(t)\| \leq [(1+2\|C\|)\|\varphi\| + \frac{\|f\|t^\alpha}{\Gamma(\alpha+1)}]e^{\frac{b_1 t^\alpha}{\Gamma(\alpha+1)}}, \quad 0 \leq t \leq n\tau \leq T.$$

According to the above, the mentioned claim is true for $n = 1$. Assume that it is true for $n = 1, \cdots, k$ (the induction hypothesis). Then using this hypothesis, it should be shown that it is satisfied for $n = k+1$ as well. Indeed, if $\tau \leq t \leq (k+1)\tau \leq T$, then

$$\|y(t)\| \leq [(1+2\|C\|)y(t-\tau) + \frac{\|f\|\tau^\alpha}{\Gamma(\alpha+1)}]e^{\frac{(\|A\|+\|B\|)\tau^\alpha}{\Gamma(\alpha+1)}}$$

and

$$\|y(t-\tau)\| \leq [(1+2\|C\|)\|\varphi\| + \frac{\|f\|(t-\tau)^\alpha}{\Gamma(\alpha+1)}]e^{\frac{b_1(t-\tau)^\alpha}{\Gamma(\alpha+1)}}.$$

Therefore,

$$\begin{aligned}\|y(t)\| & \leq \{(1+2\|C\|)[(1+2\|C\|)\|\varphi\| + \frac{\|f\|(t-\tau)^\alpha}{\Gamma(\alpha+1)}]e^{\frac{b_1(t-\tau)^\alpha}{\Gamma(\alpha+1)}} + \frac{\|f\|\tau^\alpha}{\Gamma(\alpha+1)}\}e^{\frac{(\|A\|+\|B\|)\tau^\alpha}{\Gamma(\alpha+1)}} \\ & = [(1+2\|C\|)^2\|\varphi\|e^{\frac{b_1(t-\tau)^\alpha}{\Gamma(\alpha+1)}} + (1+2\|C\|)\frac{\|f\|(t-\tau)^\alpha}{\Gamma(\alpha+1)}e^{\frac{b_1(t-\tau)^\alpha}{\Gamma(\alpha+1)}} + \frac{\|f\|\tau^\alpha}{\Gamma(\alpha+1)}]e^{\frac{(\|A\|+\|B\|)\tau^\alpha}{\Gamma(\alpha+1)}} \\ & \leq [(1+2\|C\|)\|\varphi\| + \frac{\|f\|t^\alpha}{\Gamma(\alpha+1)}]e^{\frac{b_1 t^\alpha}{\Gamma(\alpha+1)}}.\end{aligned}$$

That is

$$\|x(t)\| \leq [(1+2\|C\|)\|\varphi\| + \frac{\|f\|t^\alpha}{\Gamma(\alpha+1)}]e^{\frac{b_1 t^\alpha}{\Gamma(\alpha+1)}} \leq [(1+2\|C\|)\|\varphi\| + \frac{Mt^\alpha}{\Gamma(\alpha+1)}]e^{\frac{b_1 T^\alpha}{\Gamma(\alpha+1)}}, \quad t \in [0,T].$$

Finally, using the basic condition of Theorem 3.4, it follows:

$$\|x(t)\| < \varepsilon, \quad t \in [0,T].$$

This prove the theorem.

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分数阶中立型微分方程的有限时间稳定

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摘要: 本文研究了一类分数阶中立型微分方程的有限时间稳定性问题. 利用分步法及微分不等式理论, 获得了该方程解的存在唯一性及有限时间稳定性结果.推广了文献[8, 9]的相关结果.

关键词: 存在唯一; 有限时间稳定; 分数阶中立型

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