Vol. 34 (2014)
No. 1

数 学 杂 志
J. of Math. (PRC)

A CLASS OF IDENTITY PRODUCT DETERMINED JORDAN ALGEBRAS

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Abstract: In this paper, we investigate the condition and classification of the identity product determined Jordan matrix algebras $\mathcal{M} = M_n(R)$. Using the base matrix and the symmetric bilinear map $\{\cdot,\cdot\}$ skillfully constructed for this purpose and expansion, only elementary matrix method is used. Comparing to the reference [1], we obtain a new series of equally important definition, conclusions and proof improving the conclusions of the reference [1]. As an application we characterize the invertible linear maps on $\mathcal{M}$ which preserve identity (Jordan) product.

Keywords: bilinear maps; zero product determined algebras; identity product determined algebras

2010 MR Subject Classification: 15A03; 15A04; 15A27; 15A99

Document code: A **Article ID:** 0255-7797(2014)01-0025-06

1 Introduction

The concept of zero product determined associative (resp., Lie, Jordan) algebras was recently introduced by Brešar et al. in [1] and was further studied in [3–7]. The original motivation for introducing these concepts emerges from the discovery that certain problems concerning linear maps on algebras, such as describing linear maps preserving commutativity or zero products, can be effectively treated by first examining bilinear maps satisfying certain related conditions (see [2] for details).

Let $\mathcal{A}$ be an algebra (associative or not) over a commutative ring R with e the identity element. $\mathcal{A}$ is called to be zero product determined (see [1]) if for every R-module X and for every bilinear map $\{\cdot,\cdot\} : \mathcal{A}\times\mathcal{A}\to X$, the following two conditions are equivalent:

(i) $\{x, y\} = 0$ whenever $xy = 0$;

(ii) there exists a linear map $f : \mathcal{A}^2 \to X$ such that $\{x, y\} = f(xy)$ for any $x, y \in \mathcal{A}$.

In [1], Brešar and others proved that the full matrix algebra $M_n(R)$ is always zero product determined and zero Lie product determined. If 2 is invertible in R and $n \geq 3$ then $M_n(R)$ is also zero Jordan product determined (see [1] for details). In [3], Grašič showed that the Lie algebra of all $n\times n$ skew-symmetric matrices over an arbitrary field F of characteristic not 2 is zero Lie product determined, as is the simple Lie algebra of the symplectic type over

* **Received date:** 2011-12-22 **Accepted date:** 2012-10-23

Foundation item: Supported by the Fundamental Research Funds for the Central Universities (2010LKSX05); the National Natural Science Foundation of China (11171343).

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the field F. In [4], Grašič proved that the Jordan algebra of symmetric matrices with respect to either transpose or symplectic involution is zero product determined. In [5], the main result of [3] was extended to more general cases. It was shown in [5] that every parabolic subalgebra of a (finite-dimensional) simple Lie algebra defined over an algebraically closed field is always zero product determined. In [6], Ma and others introduced a more general concept called square-zero determined algebra and proved that the algebra of all strictly upper triangular matrices over a unital commutative ring with 2 being invertible is such an algebra. Let $z \in \mathcal{M}$ be a fixed rank one matrix and e the identity matrix. In [7], Long Wang and others proved that if a bilinear map $\{\cdot,\cdot\}$ from $\mathcal{M}\times\mathcal{M}$ to an R-module X satisfies the condition that $\{x,y\}=\{e,z\}$ whenever $xy=z$, then there exists a linear map $f:\mathcal{M}\to\mathcal{M}$ such that $\{x,y\}=f(xy)$ for any $x,y\in\mathcal{M}$. In matrix theory, besides zero product of elements, identity product of elements also plays important role. Inspired by above articles, we now introduce a new concept as follows.

Definition 1.1 Let $\mathcal{A}$ be an algebra (associative or not) over a commutative ring R with e the identity element. We say that $\mathcal{A}$ is identity product determined if for every R-module X and for every symmetric bilinear map $\{\cdot,\cdot\}:\mathcal{A}\times\mathcal{A}\to X$, the following two conditions are equivalent:

(i) $\{x,y\}=\{u,v\}$ whenever $xy=uv=e$;

(ii) $\{x,y\}=\{u,v\}$ for any pairs $x,y\in\mathcal{A}$ and $u,v\in\mathcal{A}$ satisfying $xy=uv$.

Condition (ii) is equivalent to the following condition.

(iii) There exists a linear map f from $\mathcal{A}^2$ to X such that $\{x,y\}=f(xy)$ for all $x,y\in\mathcal{A}$, where $\mathcal{A}^2$ means the algebra spanned by all elements xy.

In fact, (iii) obviously implies (ii). Conversely, if (ii) holds, we define f as $f(x)=\{x,e\}$, then f is linear and $\{x,y\}=\{xy,e\}=f(xy)$ for all $x,y\in\mathcal{A}$. Thus (iii) holds. In this paper we obtain a result as follows.

Theorem 1.2 Let R be a commutative ring such that $6\in R$ is invertible. Then the Jordan algebra $\mathcal{M}=M_n(R)$ with the Jordan product $x\circ y=\frac{xy+yx}{2}$ is identity product determined.

Remark As an associative algebra with the ordinary product, $M_n(R)$ is not identity product determined. To show this, we take X to be $M_n(R)$ itself and define the symmetric bilinear map $\{\cdot,\cdot\}$ as $\{x,y\}=\frac{1}{2}(xy+yx)$. If $xy=uv=e$, then we have $\{x,y\}=\{u,v\}=e$. However, $\{e_{12},e_{21}\}\neq\{e_{11},e_{11}\}$, although $e_{12}e_{21}=e_{11}e_{11}$, where e_{ij} denotes the matrix unit which has 1 at the (i,j) position and 0 elsewhere, e denotes the identity matrix.

2 Proof of Main Result

For the proof of Theorem 1.2, we need give a lemma firstly.

Lemma 2.1 Let R be a commutative ring with $6\in R$ being invertible, $\mathcal{M}=M_n(R)$, and X be an R-module. For $x,y\in\mathcal{M}$, assume that $x\circ y=\frac{xy+yx}{2}$ is the Jordan product. Let $\{\cdot,\cdot\}$ be a symmetric bilinear map from $\mathcal{M}\times\mathcal{M}$ to an R-module X such that $\{x,y\}=\{u,v\}$ whenever $x\circ y=u\circ v=e$. Then $\{e_{ij},e_{kl}\}=\{e_{ij}\circ e_{kl},e\}$ for each pair matrix units e_{ij} and

e_{kl}.

Proof We divide the proof into several cases.

Case 1 We show that $\{e_{ii}, e_{ii}\} = \{e_{ii} \circ e_{ii}, e\}$ for $i = 1, 2, \ldots, n$, as follows.

By $(e + e_{ii}) \circ \left(e - \frac{1}{2}e_{ii}\right) = e \circ e = e$, we have

$$\{e + e_{ii}, e - \frac{1}{2}e_{ii}\} = \{e, e\}, \tag{2.1}$$

which implies that

$$\{e_{ii}, e_{ii}\} = \{e_{ii}, e\} = \{e_{ii} \circ e_{ii}, e\}. \tag{2.2}$$

Case 2 We show that $\{e_{ij}, e_{ij}\} = \{e_{ij} \circ e_{ij}, e\}$ for $i \neq j$, as follows.

By $(e + e_{ij}) \circ (e - e_{ij}) = e \circ e = e$, we have

$$\{e + e_{ij}, e - e_{ij}\} = \{e, e\} \tag{2.3}$$

which implies that $\{e_{ij}, e_{ij}\} = 0$. Also, $\{e_{ij} \circ e_{ij}, e\} = 0$. Thus the result follows.

Case 3 We show that$\{e_{ij}, e_{kl}\} = \{e_{ij} \circ e_{kl}, e\}$ for $i \neq j$, $i \neq l$, $k \neq j$, $k \neq l$, as follows.

In this case, since $(e + e_{ij} + e_{kl}) \circ (e - e_{ij} - e_{kl}) = e \circ e = e$ we have

$$\{e + e_{ij} + e_{kl}, e - e_{ij} - e_{kl}\} = \{e, e\}. \tag{2.4}$$

Recalling the facts that $\{e_{ij}, e_{ij}\} = \{e_{kl}, e_{kl}\} = 0$ we have that $\{e_{ij}, e_{kl}\} = 0$. Also, $\{e_{ij} \circ e_{kl}, e\} = 0$. Thus the result follows.

Case 4 We show that $\{e_{ii}, e_{jj}\} = \{e_{ii} \circ e_{jj}, e\}$ for $i \neq j$, as follows.

Since $(e + e_{ii} + e_{jj}) \circ (e - \frac{1}{2}e_{ii} - \frac{1}{2}e_{jj}) = e$ we have that

$$\{e + e_{ii} + e_{jj}, e - \frac{1}{2}e_{ii} - \frac{1}{2}e_{jj}\} = \{e, e\}. \tag{2.5}$$

Recalling the facts that $\{e_{ii}, e_{ii}\} = \{e_{ii}, e\}$ and $\{e_{jj}, e_{jj}\} = \{e_{jj}, e\}$ we have $\{e_{ii}, e_{jj}\} = 0$.

On the other hand, $\{e_{ii} \circ e_{jj}, e\} = 0$. Thus the result holds.

Case 5 We show that $\{e_{ii}, e_{kl}\} = \{e_{ii} \circ e_{kl}, e\}$ for $i \neq k$, $i \neq l$ and $k \neq l$, as follows.

In this case, by $(e + e_{ii} + e_{kl}) \circ (e - \frac{1}{2}e_{ii} - e_{kl}) = e$ we have

$$\{e + e_{ii} + e_{kl}, e - \frac{1}{2}e_{ii} - e_{kl}\} = \{e, e\}. \tag{2.6}$$

Using the known results that $\{e_{ii}, e_{ii}\} = \{e_{ii}, e\}$ and $\{e_{kl}, e_{kl}\} = 0$, we have that $\frac{3}{2}\{e_{ii}, e_{kl}\} = 0$, which implies that $\{e_{ii}, e_{kl}\} = 0$(recalling that 6 is invertible). Thus, $\{e_{ii}, e_{kl}\} = \{e_{ii} \circ e_{kl}, e\}$ since $\{e_{ii} \circ e_{kl}, e\} = 0$.

Case 6 We show that $\{e_{ij}, e_{il}\} = \{e_{ij} \circ e_{il}, e\}$ for $i \neq j$, $i \neq l$ and $j \neq l$, as follows.

In this case, since $(e + e_{ij} + e_{il}) \circ (e - e_{ij} - e_{il}) = e$ we have

$$\{e + e_{ij} + e_{il}, e - e_{ij} - e_{il}\} = \{e, e\}. \tag{2.7}$$

Using the known results that $\{e_{ij}, e_{ij}\} = \{e_{il}, e_{il}\} = 0$, we have that $\{e_{ij}, e_{il}\} = 0$. Thus,

$$\{e_{ij}, e_{il}\} = \{e_{ij} \circ e_{il} e\}, \tag{2.8}$$

since $\{e_{ij} \circ e_{il}, e\} = 0$.

Case 7 We show that $\{e_{il}, e_{jl}\} = \{e_{il} \circ e_{jl}, e\}$ for $i \neq j$, $i \neq l$ and $j \neq l$, as follows. In this case, since $(e + e_{il} + e_{jl}) \circ (e - e_{il} - e_{jl}) = e$ we have

$$\{e + e_{il} + e_{jl}, e - e_{il} - e_{jl}\} = \{e, e\}. \tag{2.9}$$

Using the known results that $\{e_{il}, e_{il}\} = \{e_{jl}, e_{jl}\} = 0$, we have that $\{e_{il}, e_{jl}\} = 0$. Thus,

$$\{e_{il}, e_{jl}\} = \{e_{il} \circ e_{jl} e\}, \tag{2.10}$$

since $\{e_{il} \circ e_{jl}, e\} = 0$.

Case 8 We show that $\{e_{ii}, e_{ij}\} = \{e_{ii} \circ e_{ij}, e\}$ for $i \neq j$, as follows.

In this case, by $(e + e_{ii} + e_{ij}) \circ (e - \frac{1}{2}e_{ii} - \frac{1}{2}e_{ij}) = e$ we have that

$$\{e + e_{ii} + e_{ij}, e - \frac{1}{2}e_{ii} - \frac{1}{2}e_{ij}\} = \{e, e\}. \tag{2.11}$$

Using known results that $\{e_{ii}, e_{ii}\} = \{e_{ii} \circ e_{ii}, e\} = \{e_{ii}, e\}$ and $\{e_{ij}, e_{ij}\} = 0$ we obtain

$$\{e_{ii}, e_{ij}\} = \frac{1}{2}\{e_{ij}, e\}. \tag{2.12}$$

This leads to the required result.

Case 9 We show that $\{e_{jj}, e_{ij}\} = \{e_{jj} \circ e_{ij}, e\}$ for $i \neq j$, as follows.

In this case, by $(e + e_{jj} + e_{ij}) \circ (e - \frac{1}{2}e_{jj} - \frac{1}{2}e_{ij}) = e$ we have that

$$\{e + e_{jj} + e_{ij}, e - \frac{1}{2}e_{jj} - \frac{1}{2}e_{ij}\} = \{e, e\}. \tag{2.13}$$

Using known results that $\{e_{jj}, e_{jj}\} = \{e_{jj} \circ e_{jj}, e\} = \{e_{jj}, e\}$ and $\{e_{ij}, e_{ij}\} = 0$ we obtain

$$\{e_{jj}, e_{ij}\} = \frac{1}{2}\{e_{ij}, e\}. \tag{2.14}$$

This leads to the required result.

Case 10 We show that $\{e_{ij}, e_{jl}\} = \{e_{ij} \circ e_{jl}, e\}$ for $i \neq j$, $j \neq l$ and $i \neq l$, as follows.

In this case, by $(e + e_{ij} + e_{jl}) \circ (e - e_{ij} - e_{jl} + e_{il}) = e$ we have that

$$\{e + e_{ij} + e_{il}, e - e_{ij} - e_{jl} + e_{il}\} = \{e, e\}. \tag{2.15}$$

Using the known facts that $\{e_{ij}, e_{ij}\} = \{e_{jl}, e_{jl}\} = \{e_{il}, e_{il}\} = \{e_{ij}, e_{il}\} = \{e_{il}, e_{jl}\} = 0$, we have that

$$2\{e_{ij}, e_{jl}\} = \{e, e_{il}\}. \tag{2.16}$$

This leads to the desired result.

Case 11 We show that $\{e_{ij}, e_{ji}\} = \{e_{ij} \circ e_{ji}, e\}$ for $i \neq j$, as follows.

In this case, by $(e - e_{ii} - e_{jj} + e_{ij} + e_{ji}) \circ (e - e_{ii} - e_{jj} + e_{ij} + e_{ji}) = e$ we have that

$$\{e - e_{ii} - e_{jj} + e_{ij} + e_{ji}, e - e_{ii} - e_{jj} + e_{ij} + e_{ji}\} = \{e, e\}. \tag{2.17}$$

Recalling that $\{e_{ii}, e_{ii}\} = \{e, e_{ii}\}$, $\{e_{jj}, e_{jj}\} = \{e, e_{jj}\}$, $\{e_{ij}, e_{ij}\} = \{e_{ji}, e_{ji}\} = 0$, $\{e_{ii}, e_{ij}\} = \{e_{jj}, e_{ij}\} = \frac{1}{2}\{e_{ij}, e\}$ and $\{e_{ii}, e_{ji}\} = \{e_{jj}, e_{ji}\} = \frac{1}{2}\{e_{ji}, e\}$, we have that

$$2\{e_{ij}, e_{ji}\} = \{e_{ii}, e\} + \{e_{jj}, e\}. \tag{2.18}$$

This follows that $\{e_{ij}, e_{ji}\} = \{e_{ij} \circ e_{ji}, e\}$.

Combining Case 1–Case 11 we conclude that $\{e_{ij}, e_{kl}\} = \{e_{ij} \circ e_{kl}, e\}$ for each pair e_{ij} and e_{kl}.

Proof of Theorem 1.2 Let $\{\cdot, \cdot\}$ be a symmetric bilinear map from $\mathcal{M} \times \mathcal{M}$ to an R-module X such that $\{x, y\} = \{u, v\}$ whenever $x \circ y = u \circ v = e$. It follows from Lemma 2.1 that

$$\{e_{ij}, e_{kl}\} = \{e_{ij} \circ e_{kl}, e\}$$

for each pair of matrix units e_{ij} and e_{kl} in $\mathcal{M}$. We now show that $\{x, y\} = \{x \circ y, e\}$ for all $x, y \in \mathcal{M}$. Assume that $x = [x_{ij}]_{n\times n},\ y = [y_{kl}]_{n\times n} \in M_n(R)$, express both of them as the linear combinations of the matrix units

$$x = \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}e_{ij},\ \ y = \sum_{k=1}^{n}\sum_{l=1}^{n} y_{kl}e_{kl}.$$

Since $\{\cdot, \cdot\}$ is bilinear we have that

$$\{x, y\} = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} x_{ij}y_{kl}\{e_{ij}, e_{kl}\} = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} x_{ij}y_{kl}\{e_{ij} \circ e_{kl}, e\} = \{x \circ y, e\}.$$

Thus, if $x \circ y = u \circ v$, then $\{x, y\} = \{x \circ y, e\} = \{u \circ v, e\} = \{u, v\}$. This completes the proof.

3 Application of Main Result

In the last decade considerable works have been done concerning characterization of maps on matrix algebras or operator algebras which preserves zero product (see [8–11]). Since identity product of elements also plays important role in operator theory we now characterize the invertible linear maps on the Jordan algebra $\mathcal{M}$ which preserve identity (Jordan) product. We find that by applying Theorem 1.2 this problem is easy to solve.

Corollary 3.1 Assume that the Jordan algebra $\mathcal{M}$ and the ring R are as in Theorem 1.2. Let ϕ be an invertible linear map on $\mathcal{M}$ and such that ϕ fixes e. Then ϕ preserves identity (Jordan) product, i.e., $\phi(x) \circ \phi(y) = e \ \Leftrightarrow x \circ y = e$ if and only if ϕ is a Jordan automorphism of $\mathcal{M}$, i.e., $\phi(x \circ y) = \phi(x) \circ \phi(y)$ for all $x, y \in \mathcal{M}$.

Proof The sufficient condition is obvious. Now we consider the necessary condition. Suppose that ϕ preserves identity (Jordan) product. We define $\{\cdot, \cdot\}$ as $\{x, y\} = \phi(x) \circ \phi(y)$ for $x, y \in \mathcal{M}$. Then it is a symmetric bilinear map, and if $a \circ b = c \circ d = e$, then

$$\{a, b\} = \phi(a) \circ \phi(b) = e = \phi(c) \circ \phi(d) = \{c, d\}.$$

By Theorem 1.2, we know $\{x, y\} = \{u, v\}$ whenever $x \circ y = u \circ v$. Now for any $x, y \in \mathcal{M}$, since $x \circ y = (x \circ y) \circ e$ we have

$$\phi(x) \circ \phi(y) = \{x, y\} = \{x \circ y, e\} = \phi(x \circ y) \circ \phi(e) = \phi(x \circ y).$$

This shows that ϕ is a Jordan automorphism.

Remark The invertible linear maps on the associative algebra $M_n(R)$ preserving identity (ordinary) product were described in [12] by using a much direct method. Comparing with Theorem 2.1 in [12], we find that the proof of above corollary is much easy.

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单位积决定的若当矩阵代数

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摘要: 本文研究了单位积决定的若当矩阵代数$\mathcal{M} = M_n(R)$ 的条件及分类问题. 利用基矩阵及巧妙对对称双线性映射$\{\cdot, \cdot\}$ 进行构造和扩充, 用初等矩阵的方法, 获得了一系列新的同样重要的定义, 结论与证明(与参考文献 [1]相比较), 推广了参考文献 [1]的结论, 作为其应用可以进一步证明了$M_n(R)$ 上的任意可逆线性映射都是保单位积的.

关键词: 双线性映射; 零积决定的代数; 单位积决定的代数

MR(2010)**主题分类号**: 15A03; 15A04; 15A27; 15A99 **中图分类号**: O151.2