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ON THE GROWTH OF SOLUTIONS OF HIGHER-ORDER ALGEBRAIC DIFFERENTIAL EQUATIONS

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Abstract: This paper investigates the problem of the growth of solution of higher-order algebraic differential equations. Using the Nevanlinna value distribution theory of meromorphic functions and some skills of differential equations theory, we obtain a result which is more precise and more general, and extend the theories of He and Laine .

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1 Introduction and Main Result

In what follows, we assume the reader is familiar with the standard notions of Nevanlinna's value distribution theory as the proximity function $m(r, w)$, the integrated counting function $N(r, w)$, the characteristic function $T(r, w)$, see e.g. [1, 2]. Many authors investigated the algebraic differential equations and obtained many results (see [3–10]).

An analytic function $w(z)$ with ν branches is an algebroid function if the function $w(z)$ satisfies an equation of the form

$$\psi(z, w) = A_v(z)w^v + A_{v-1}(z)w^{v-1} + \cdots + A_0(z),$$

where $A_j(z)$ $(j = 0, 1, \cdots, v)$ are regular functions with no commom zeros and $A_\nu \neq 0$, especially, when $\nu = 1$, $w(z)$ be a meromorphic function; when $A_j(z)(j = 0, 1, \cdots, v)$ are polynomials, $w(z)$ is an algebraic function. Some notations

$$N_x(r, w) = \frac{1}{\nu}\int_0^r \frac{n_x(t, w) - n_x(0, w)}{t}dt + \frac{1}{\nu}n_x(0, w)\log r,$$

and Toda gave the definition of $N_b(r, w)$.

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Let $w = w(z)$ be a ν-valued algebroid function and a be a pole of w. Then, in the neighbourhood of a, we have the following expansions of w:

$$w(z) = (z-a)^{-\tau_i/\lambda_i} S((z-a)^{1/\lambda_i}),$$

where $i = 1, 2, \cdots, \mu(a)(\leq \nu)$, $1 \leq \tau_i$, $1 \leq \lambda_i$, $\sum \lambda_i = \nu$ and $S(t)$ is a regular power series of t such that $S(0) \neq 0$. Put

$$n_b(r, w) = \sum_{|a| \leq r} \sum_{i=1}^{\mu(a)} (\lambda_i - 1)$$

and

$$N_b(r, w) = \frac{1}{\nu} \int_0^r (n_b(t, w) - n_b(0, w))/t dt + \frac{1}{\nu} n_b(0, w) \log r.$$

It is trivial that $N_b(r, w) \leq (\nu - 1)\overline{N}(r, w)$.

About the differential equation

$$\sum_{(i) \in I} a_{(i)}(z) w^{i_0} (w')^{i_1} \cdots (w^{(n)})^{i_n} = \frac{\sum\limits_{i=0}^{p} a_i(z) w^i}{\sum\limits_{j=0}^{q} b_j(z) w^j}, \tag{1.1}$$

where $a_{(i)}(z)$, $a_i(z)$ and $b_j(z)$ are meromorphic coefficients. He and Laine investigated the problem of the growth of solutions of it and obtain the following result.

Theorem A [2] Let $w(z)$ be an algebroid solution of (1.1) with ν branches and $p > q+\lambda$. Then for any $\xi > 1$, there exist a positive constant K and r_0 such that for all $r \geq r_0$, we have $T(r, w) \leq KF(\xi r)$, where

$$F(r) = \overline{N}(r, w) + \sum_{(i)} T(r, a_{(i)}) + \sum_{i=0}^{p} T(r, a_i) + \sum_{j=0}^{q} T(r, b_j) + 1.$$

In this paper, we discuss the problem of the growth of solutions of generalized higher-order algebraic differential equation of the form

$$\frac{\Omega_1(z, w)}{\Omega_2(z, w)(w-a)^\lambda} = \frac{\sum\limits_{i=0}^{p} a_i(z) w^i}{\sum\limits_{j=0}^{q} b_j(z) w^j}, \tag{1.2}$$

where $\Omega_1(z, w)$ and $\Omega_2(z, w)$ be two differential polynomials, a be a nonzero complex constant, and

$$\Omega_1(z, w) = \sum_{(i) \in I} a_{(i)}(z) w^{i_0} (w')^{i_1} \cdots (w^{(n)})^{i_n} (n \geq 1),$$

$$\Omega_2(z, w) = \sum_{(j) \in J} b_{(j)}(z) w^{j_0} (w')^{j_1} \cdots (w^{(m)})^{j_m} (m \geq 1),$$

other notations

$$\lambda_1=\max_{(i\in I)}\{\sum_{\alpha=0}^{n} i_\alpha\},\lambda_2=\max_{(j\in J)}\{\sum_{\beta=0}^{m} j_\beta\},\ \lambda=\max\{\lambda_1,\lambda_2\},$$

$$\overline{\mu}_1=\max_{(i\in I)}\{\sum_{\alpha=0}^{n} \alpha i_\alpha\},\overline{\mu}_2=\max_{(j\in J)}\{\sum_{\beta=0}^{m} \beta j_\beta\},\ \overline{\mu}=\max\{\overline{\mu}_1,\overline{\mu}_2\},$$

$$\Delta_1=\max_{(i\in I)}\{\sum_{\alpha=0}^{n}(\alpha+1)i_\alpha\},\ \ \Delta_2=\max_{(j\in J)}\{\sum_{\beta=0}^{m}(\beta+1)j_\beta\},\ \ \Delta=\max\{\Delta_1,\Delta_2\},$$

$$\sigma_1=\max_{(i\in I)}\{\sum_{\alpha=0}^{n}(2\alpha-1)i_\alpha\},\ \ \sigma_2=\max_{(j\in J)}\{\sum_{\beta=0}^{m}(2\beta-1)j_\beta\},\ \ \sigma=\max\{\sigma_1,\sigma_2\},$$

$$l_1=\max_{(i\in I)}\{\sum_{\alpha=0}^{n}(\alpha-1)i_\alpha\},\ \ l_2=\max_{(j\in J)}\{\sum_{\beta=0}^{m}(\beta-1)j_\beta\},\ \ l=\max\{l_1,l_2\},$$

and obtain the following result.

Theorem 1 Let $w(z)$ be an algebroid solution of (1.2) with ν branches and $p>q+2\lambda$. Then for any $\xi>1$, there exist a positive constant K and r_0 such that for all $r\geq r_0$, we get $T(r,w)\leq KF(\xi r)$, where

$$\begin{aligned} F(r) &= \overline{N}(r,w)+N_x(r,w)+N_b(r,w)\\ &\quad+\sum_{(i)}T(r,a_{(i)})+\sum_{(j)}T(r,b_{(j)})+\sum_{i=0}^{p}T(r,a_i)+\sum_{j=0}^{q}T(r,b_j)+1,\\ \lambda &= \max\{\lambda_1,\lambda_2\}. \end{aligned}$$

2 Some Lemmas

Lemma 1 [1] Let $R(z,w)=\frac{\sum_{i=0}^{p}a_i(z)w^i}{\sum_{j=0}^{q}b_j(z)w^j}$ be an irreducible rational function in $w(z)$ with the meromorphic coefficients $a_i(z)$ and $b_j(z)$. If $w(z)$ is an algebroid function, then

$$T(r,R(z,w))=\max\{p,q\}T(r,w)+O\{\sum T(r,a_i)+\sum T(r,b_j)\}.$$

Lemma 2 Let $w(z)$ be an algebroid function, and $\Omega(z,w)$ be as in (1.2), a be a nonzero complex constant. Then

$$T(r,\frac{\Omega(z,w)}{(w-a)^\lambda})\leq \lambda T(r,w)+\overline{\mu}[\overline{N}(r,w)+N_b(r,w)]+\sum_{(i)}T(r,a_{(i)})+\sum_{\alpha=1}^{n}m(r,\frac{w^{(\alpha)}}{w-a})+O(1).$$

Proof Let w be an algebroid function with ν branches, we denote by L a curve connecting all branch points of $w(z)$ and $C'=C\backslash L$, then every branch $w_j(z)(j=1,2,\cdots,\nu)$ of

$w(z)$ is single-valued in C'. Set $E=\{z,|z|=r\}$, $E_1^j=\{z\epsilon E,|w_j(z)-a|\geq 1\}$, $E_2^j=E\backslash E_1^j$, and $z=re^{i\theta}$, so that

$$\frac{1}{2\pi}\int_E \log^+|\frac{\Omega(z,w_j)}{(w_j-a)^\lambda}|d\theta=\frac{1}{2\pi}(\int_{E_1^j}+\int_{E_2^j})\log^+|\frac{\Omega(z,w_j)}{(w_j-a)^\lambda}|d\theta.$$

Set $\lambda_i=i_0+i_1+\cdots+i_n$, when $z\epsilon E_1^j$, it is easy to show

$$\begin{aligned}|\frac{\Omega(z,w_j)}{(w_j-a)^\lambda}| &= |\frac{\sum\limits_{(i)}a_{(i)}(z)(\frac{(w_j-a)'}{w_j-a})^{i_1}\cdots(\frac{(w_j-a)^{(n)}}{w_j-a})^{i_n}}{(w_j-a)^{\lambda-\lambda_i}}|\\ &\leq \sum_{(i)}|a_{(i)}(z)||(\frac{(w_j-a)'}{w_j-a})^{i_1}|\cdots|(\frac{(w_j-a)^{(n)}}{w_j-a})^{i_n}|,\end{aligned}$$

by the lemma of Logarithmic Derivate, we have

$$\begin{aligned}&\frac{1}{2\pi}\int_{E_1^j}\log^+|\frac{\Omega(z,w_j)}{(w_j-a)^\lambda}|d\theta\leq\frac{1}{2\pi}\int_{E_1^j}\log^+\sum_{(i)}|a_{(i)}(z)|d\theta+\sum_{\alpha=1}^n m(r,\frac{(w_j-a)^{(\alpha)}}{w_j-a})\\ =&\sum_{(i)}m(r,a_{(i)})+\sum_{\alpha=1}^n m(r,\frac{w_j^{(\alpha)}}{w_j-a});\end{aligned}$$

when $z\epsilon E_2^j$, we have

$$\begin{aligned}&|\frac{\Omega(z,w_j)}{(w_j-a)^\lambda}|=|\frac{\sum\limits_{(i)}a_{(i)}(z)(\frac{(w_j-a)'}{w_j-a})^{i_1}\cdots(\frac{(w_j-a)^{(n)}}{w_j-a})^{i_n}}{(w_j-a)^{\lambda-\lambda_i}}|\\ \leq\ &|\frac{1}{(w_j-a)^{\lambda-\lambda_i}}|\sum_{(i)}|a_{(i)}(z)||(\frac{(w_j-a)'}{w_j-a})^{i_1}|\cdots|(\frac{(w_j-a)^{(n)}}{w_j-a})^{i_n}|,\end{aligned}$$

then

$$\begin{aligned}&\frac{1}{2\pi}\int_{E_2^j}\log^+|\frac{\Omega(z,w_j)}{(w_j-a)^\lambda}|d\theta\\ \leq\ &\frac{1}{2\pi}\int_{E_2^j}\log^+\sum_{(i)}|a_{(i)}(z)|d\theta+\frac{\lambda}{2\pi}\int_{E_2^j}\log^+|\frac{1}{w_j-a}|d\theta+\sum_{\alpha=1}^n m(r,\frac{(w_j-a)^{(\alpha)}}{w_j-a})\\ =\ &\sum_{(i)}m(r,a_{(i)}(z))+\lambda m(r,\frac{1}{w_j-a})+\sum_{\alpha=1}^n m(r,\frac{w_j^{(\alpha)}}{w_j-a}).\end{aligned}$$

Hence we obtain

$$\begin{aligned}m(r,\frac{\Omega(z,w)}{(w-a)^\lambda})&=\frac{1}{\nu}\sum_{j=1}^{\nu}\frac{1}{2\pi}(\int_{E_1^j}+\int_{E_2^j})\log^+|\frac{\Omega(z,w_j)}{(w_j-a)^\lambda}|d\theta\\ &\leq\ \lambda m(r,\frac{1}{w-a})+\sum_{(i)}m(r,a_{(i)}(z))+\sum_{\alpha=1}^n m(r,\frac{w^{(\alpha)}}{w-a}).\end{aligned}\tag{2.1}$$

Next, we estimate the poles of $\frac{\Omega(z,w)}{(w-a)^\lambda}$, we denote by $\tau(z_0, f)$ the order of pole of f at $z = z_0$. Now we discuss the following two cases.

(i) When z_0 is not a pole of $w(z)$, we have

$$\begin{aligned}\tau(z_0, \frac{\Omega(z,w)}{(w-a)^\lambda}) &\leq \tau(z_o, \frac{1}{(w-a)^\lambda}) + \sum \tau(z_0, a_{(i)}(z)) \\ &= \lambda\tau(z_0, \frac{1}{w-a}) + \sum \tau(z_0, a_{(i)}(z)). \end{aligned} \tag{2.2}$$

(ii) When z_0 is a pole of $w(z)$, in a neighbourhood of z_0,

$$\begin{aligned}&w - a = (z - z_0)^{-\tau/\beta} S((z - z_0)^{1/\beta}) \ \ (\beta \geq 1, \tau \geq 1), \\ &w^{(\alpha)} = (w - a)^{(\alpha)} = (z - z_0)^{(-\tau+\beta\alpha)/\beta} S_\alpha(z), S_\alpha(z_0) \neq 0, \infty,\end{aligned}$$

then

$$\tau(z_0, (\frac{w^{(\alpha)}}{w-a})^{i_\alpha}) = \tau(z_0, (\frac{(w-a)^{(\alpha)}}{w-a})^{i_\alpha}) = \beta\alpha i_\alpha.$$

Set $a_{(i)}(z) w^{i_0} (w')^{i_1} \cdots (w^n)^{i_n}$ is a general term of $\Omega(z, w(z))$, we obtain

$$\tau(z_0, a_{(i)}(z) w^{i_0} (w')^{i_1} \cdots (w^{(n)})^{i_n} / (w-a)^\lambda) \leq \beta \sum_{a=1}^{n} \alpha i_\alpha + \tau(z_0, a_{(i)}(z)),$$

then

$$\begin{aligned}\tau(z_0, \frac{\Omega(z,w)}{(w-a)^\lambda}) &\leq \beta \max\{\sum_{a=1}^{n} \alpha i_\alpha\} + \tau(z_0, a_{(i)}(z)) = \beta\overline{\mu} + \sum \tau(z_0, a_{(i)}(z)) \\ &= \overline{\mu}(\beta - 1) + \overline{\mu} + \tau(z_0, a_{(i)}(z)). \end{aligned} \tag{2.3}$$

Combining (2.2) with (2.3) we deduce

$$N(r, \frac{\Omega(z,w)}{(w-a)^\lambda}) \leq \lambda N(r, \frac{1}{w-a}) + \overline{\mu} N_b(r, w) + \overline{\mu}\overline{N}(r, w) + \sum_{(i)} N(r, a_{(i)}(z)). \tag{2.4}$$

Combining (2.1) with (2.4) we have

$$\begin{aligned}&T(r, \frac{\Omega(z,w)}{(w-a)^\lambda}) \\ \leq\ & \lambda T(r, \frac{1}{w-a}) + \overline{\mu} N_b(r, w) + \overline{\mu}\overline{N}(r, w) + \sum_{(i)} T(r, a_{(i)(z)}) + \sum_{\alpha=1}^{n} m(r, \frac{w^{(\alpha)}}{w-a}) \\ =\ & \lambda T(r, w-a) + \overline{\mu}[N_b(r, w) + \overline{N}(r, w)] + \sum_{(i)} T(r, a_{(i)(z)}) + \sum_{\alpha=1}^{n} m(r, \frac{w^{(\alpha)}}{w-a}) + O(1) \\ =\ & \lambda T(r, w) + \overline{\mu}[N_b(r, w) + \overline{N}(r, w)] + \sum_{(i)} T(r, a_{(i)(z)}) + \sum_{\alpha=1}^{n} m(r, \frac{w^{(\alpha)}}{w-a}) + O(1).\end{aligned}$$

Lemma 3 Let $w(z)$ be an algebroid function, then we have

$$T(r,\Omega(z,w))\leq \lambda T(r,w)+\overline{\mu}\overline{N}(r,w)+\sigma N_x(r,w)-lN_b(r,w)+\sum_{(i)}T(r,a_{(i)(z)})+\sum_{\alpha=1}^{n}m(r,\frac{w^{(\alpha)}}{w}).$$

Proof Proceeding similary as in the proof the Lemma 2, we can verify the assertion.

Lemma 4 [2] Let $U(r)$, $H(r)(r\in[0,\infty))$ be two nonnegative and nondecreasing functions, $H(r)\to\infty$ as $r\to\infty$, a and b be two positive numbers,

$$H(r_0)\geq \max\{(a+b)\log 2, 2^{2+\frac{b}{a}}a(a+b)\},$$

if for all r and t, when $0<r_0\leq r<t$, satisfies

$$U(r)<a\log^+ U(t)+b\log\frac{t}{t-r}+H(r),$$

then for $0<r_0\leq r<t$, we have

$$U(r)<(a+b)\log\frac{t}{t-r}+2H(r).$$

3 Proof of Theorem 1

We discuss the following two cases.

Case 1 If $w(z)$ satisfies $\sum\limits_{i=0}^{p}a_i(z)w^i\equiv 0$, then $a_p(z)w^p=-a_{p-1}(z)w^{p-1}-\cdots-a_0(z)$. From Lemma 1, there exists a positive constant K such that

$$pT(r,w)+T(r,a_p)\leq (p-1)T(r,w)+\sum_{i=0}^{p-1}T(r,a_i(z)),$$

$$T(r,w)\leq K\sum_{i=0}^{p}T(r,a_i(z))\leq KF(r).$$

Case 2 If $\sum\limits_{i=0}^{p}a_i(z)w^i\neq 0$, we rewrite equation (1.2) as follows

$$Q(z,w)\frac{\Omega_1(z,w)}{\Omega_2(z,w)(w-a)^\lambda}=P(z,w).$$

Using Lemma 1, Lemma 2 and Lemma 3, we get

$$\begin{aligned} &T(r,Q(z,w)\frac{\Omega_1(z,w)}{\Omega_2(z,w)(w-a)^\lambda})\leq T(r,Q(z,w))+T(r,\frac{\Omega_1(z,w)}{(w-a)^\lambda})+T(r,\frac{1}{\Omega_2(z,w)})\\ \leq\ & T(r,Q(z,w)+T(r,\frac{\Omega_1(z,w)}{(w-a)^\lambda})+T(r,\Omega_2(z,w))+O(1)\\ \leq\ & (q+2\lambda)T(r,w)+2\overline{\mu}\overline{N}(r,w)+(\overline{\mu}-l)N_b(r,w)+\sigma N_x(r,w)+\sum_{\alpha=1}^{n}m(r,\frac{w^{(\alpha)}}{w-a})\\ &+\sum_{\alpha=1}^{n}m(r,\frac{w^{(\alpha)}}{w})+\sum T(r,a_{(i)})+\sum T(r,b_{(j)})+\sum_{j=0}^{q}T(r,b_j)+O(1). \end{aligned} \tag{3.1}$$

By means of Lemma 1, we obtain

$$T(r,P(z,w)) = pT(r,w) + O\{\sum_{i=0}^{p} T(r,a_i)\}. \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$\begin{aligned} & pT(r,w) \\ < \quad & (q+2\lambda)T(r,w) + 2\overline{\mu}\overline{N}(r,w) + (\overline{\mu}-l)N_b(r,w) + \sigma N_x(r,w) + \sum_{i=0}^{p} T(r,a_i) \\ & + \sum T(r,a_{(i)}) + \sum T(r,b_{(j)}) + \sum_{j=0}^{q} T(r,b_j) + \sum_{\alpha=1}^{n} m(r,\frac{w^{(\alpha)}}{w-a}) + \sum_{\alpha=1}^{n} m(r,\frac{w^{(\alpha)}}{w}). \end{aligned}$$

We note that $p > q+2\lambda$. Thus

$$\begin{aligned} T(r,w) \quad < \quad & \frac{2\overline{\mu}}{p-(q+2\lambda)}\overline{N}(r,w) \\ & + \frac{\overline{\mu}-l}{p-(q+2\lambda)}N_b(r,w) + \frac{\sigma}{p-(q+2\lambda)}N_x(r,w) + F_1(r) + D(r), \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} F_1(r) &= \frac{1}{p-(q+2\lambda)}\{\sum_{i=0}^{p} T(r,a_i) + \sum T(r,a_{(i)}) + \sum T(r,b_{(j)}) + \sum_{j=0}^{q} T(r,b_j)\}, \\ D(r) &= \frac{1}{p-(q+2\lambda)}\{\sum_{\alpha=1}^{n} m(r,\frac{w^{(\alpha)}}{w-a}) + \sum_{\alpha=1}^{n} m(r,\frac{w^{(\alpha)}}{w})\}. \end{aligned}$$

We apply the generalized Lemma of Logarithmic derivate to $D(r)$. Then inequality (3.3) becomes

$$T(r,w) < a\log T(t,w) + b\log\frac{t}{t-r} + H(r), \tag{3.4}$$

where a and b are constants,and

$$H(r) = \frac{2\overline{\mu}}{p-(q+2\lambda)}\overline{N}(r,w) + \frac{\overline{\mu}-l}{p-(q+2\lambda)}N_b(r,w) + \frac{\sigma}{p-(q+2\lambda)}N_x(r,w) + F_1(r).$$

Applying Lemma 4 to (3.4) and we get

$$T(r,w) < (a+b)\log\frac{t}{t-r} + 2H(t).$$

Set $t = \xi r, \xi > 1$. Then $T(r,w) \le KF(\xi r)$.

Combining Case 1 and Case 2 we complete the proof of Theorem 1.

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高阶代数微分方程解的增长级

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摘 要: 本文研究了高阶代数微分方程解的增长级的问题. 利用亚纯函数的Nevanlinna值分布理论和微分方程的一些技巧, 得到了一个更精确和更一般的结论, 推广了何育赞和Laine的一些理论.

关 键 词: 增长级; 代数体函数; 代数微分方程

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