THE DRAZIN INVERSE OF A MODIFIED MATRIX $A - CB$

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Abstract: In this paper, we study the representations of the Drazin inverse of a modified matrix $A - CB$. By the properties of the $k$-idempotent matrix and the diagonalizable matrix, we get some new representations of the Drazin inverse through weakened conditions of literature [4].

Keywords: modified matrix; Drazin inverse; Schur complement

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1 Introduction

The problem of the Drazin inverse was discussed widely in [1–12]. The Drazin inverse was used to be applied in sigular differential difference equations, Markov chains and numerical analysis in [1–3]. The Drazin inverse of the modified matrices was studied by many people [4–6], as a modified matrix can be seen as the sum of two matrices or a matrix added a perturbed element. In [4], Wei Yiming gave the expression for the Drazin inverse of $A - CB$; Liu Xifu weakened the condition of [4] and gave another expression in [5]; the Drazin inverse of $A - CD^D B$ was given in [6]. In this paper, we weaken the conditions of [4–5] and give different results.

2 Definitions and Basic Results

Definition 2.1 Let $\mathbb{C}^{n \times n}$ denote the set of $n \times n$ complex matrices. The Drazin inverse of $A \in \mathbb{C}^{n \times n}$ is the unique matrix $A^D$ satisfying the relations:

$$A^D A A^D = A^D, \quad A^D A = A A^D, \quad A^{k+1} A^D = A^k,$$

where $k$ is the smallest non-negative integer such that $\text{rank}(A^k) = \text{rank}(A^{k+1})$, i.e., $k = \text{ind}(A)$, the index of $A$. The case when $\text{ind}(A) = 1$, the Drazin inverse is called the group inverse of $A$ and it is denoted by $A^\#$. We denote by $A^\pi$ corresponding to the eigenvalue 0 that is given by $A^\pi = I - AA^D$.

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Lemma 2.2  [4] Suppose \( P = 0, Q = 0 \) and \( C(I - ZZ^D)B = 0 \). Then
\[
(A - CB)^D = A^D + KZ^DH, \tag{2.2}
\]
where denote \( K = AD, H = BA^D, Z = I - BA^DC, P = (I - AA^D)C \) and \( Q = B(I - A^D) \).

Lemma 2.3  [5] Let \( A \) be an idempotent matrix, suppose \( P = 0, \) ind\((Z) = k, \) then
\[
(A - CB)^D = A + CZ^D H - C(Z^D)^2\sum_{i=0}^{k-1} Z^i H, \tag{2.3}
\]
where denote \( K = AC, H = BA, Z = I - BAC, P = (I - A)C \) and \( Q = B(I - A), \) especially, \( Z = I - BC \) at here.

Lemma 2.4  [6] Let \( A, B, C \) and \( D \) be complex matrices, where ind\((A) = k, \) If \( A^p C = 0, \)
\( CZ^p = 0, Z^p B = 0, CD^p = 0 \) and \( D^p B = 0, \) then
\[
(A - CD^p B)^D = A^D + AD^p C Z^D BA^D - \sum_{i=0}^{k-1} (A^p + AD^p C Z^D BA^D)^{i+1} A^p C Z^D BA^i A^p, \tag{2.4}
\]
where denote the schur complement \( Z = D - BA^D C, \) furthermore, ind\((A - CD^p B) \leq \) ind\((A). \)

3 Main Theorems and Proofs

First, we definite some notation similar to the reference [4]. Let
\[
K = AD, H = BA^D, \quad \Gamma = HK, \quad Z = I - BA^DC \tag{3.1}
\]
and
\[
P = (I - AA^D)C, \quad Q = B(I - A^D). \tag{3.2}
\]

Theorem 3.1 Let \( A, B \) and \( C \) be complex matrices, where ind\((A) = k, \) If \( A^p C = 0 \)
and \( CZ^p B = 0, \) then
\[
(A - CB)^D = A^D + KZ^DH - \sum_{i=0}^{k-1} (A^D + KZ^DH)^{i+1} KZ^D BA^i A^p. \tag{3.3}
\]
furthermore, ind\((A - CB) \leq \) ind\((A). \)

Proof  Let \( X = A^D + KZ^DH - \sum_{i=0}^{k-1} (A^D + KZ^DH)^{i+1} KZ^D BA^i A^p. \) Then,
\[
(A - CB) X = (A - CB)[A^D + KZ^DH - \sum_{i=0}^{k-1} (A^D + KZ^DH)^{i+1} KZ^D BA^i A^p]
\]
\[
= (A - CB)(A^D + KZ^DH) - (A - CB)(A^D + KZ^DH) \sum_{i=0}^{k-1} (A^D + KZ^DH)^i KZ^D BA^i A^p
\]
\[
= AA^D - AA^D \sum_{i=0}^{k-1} (A^D + KZ^DH)^i KZ^D BA^i A^p
\]
\[
= AA^D - \sum_{i=0}^{k-1} (A^D + KZ^DH)^i KZ^D BA^i A^p. \tag{3.4}
\]
At the same time, we get

\[ X(A - CB) = [A^D + KZ^DH - \sum_{i=0}^{k-1} (A^D + KZ^DH)^{i+1} KZ^DBA^iA^\pi](A - CB) \]

\[ \begin{align*}
  &= (A^D + KZ^DH)(A - CB) \\
  &\quad - \sum_{i=0}^{k-1} (A^D + KZ^DH)^{i+1} KZ^DBA^iA^\pi(A - CB) \\
  &= AA^D - A^D CZ^DBA^\pi - \sum_{i=0}^{k-1} (A^D + KZ^DH)^{i+1} KZ^DBA^{i+1}A^\pi \\
  &= AA^D - \sum_{i=0}^{k-1} (A^D + KZ^DH)^i KZ^DBA^iA^\pi. \\
\end{align*} \] (3.5)

From (3.4) and (3.5) it follows that \((A - CB)X = X(A - CB)\).

Now, using (3.5) and \(A^\pi X = 0\), we obtain

\[ \begin{align*}
  (X(A - CB) - I)X &= 0, \\
\end{align*} \]

i.e., \((A - CB)X = X\).

Finally, we will prove that \((A - CB) - (A - CB)^2 X\) is a nilpotent matrix. Using \(A^\pi C = 0, CZ^\pi B = 0\), and expressions (3.4) conveniently, it can be proved that

\[ (A - CB) - (A - CB)^2 X = AA^\pi + \sum_{i=0}^{k-1} (A^D + KZ^DH)^i KZ^DBA^{i+1}A^\pi. \]

by induction on integer \(j \geq 1\), we have

\[ [(A - CB) - (A - CB)^2 X]^j = A^jA^\pi + \sum_{i=0}^{k-1} (A^D + KZ^DH)^i KZ^DBA^{i+j}A^\pi. \]

Then we get

\[ [(A - CB) - (A - CB)^2 X]^k = 0. \] (3.6)

where \(k = \text{ind}(A)\). Therefore, we get \((A - CB)^{k+1} X = (A - CB)^k X\) and \(\text{ind}(A - CB) \leq \text{ind}(A)\).

The theorem is proved completely.

**Corollary 3.2** Let \(A, B\) and \(C\) be complex matrices, where \(\text{ind}(A) = k\). If \(BA^\pi = 0\) and \(CZ^\pi B = 0\), then

\[ (A - CB)^D = A^D + KZ^DH - \sum_{i=0}^{k-1} KZ^DBA^iA^\pi(A^D + KZ^DH)^{i+1}, \] (3.7)

Furthermore, \(\text{ind}(A - CB) \leq \text{ind}(A)\).

**Proof** The proof is similar to Theorem 3.1.
In the reference [5], it was discussed the Drazin inverse of a modified matrix \( A - CB \), where \( A \) is an idempotent matrix (Lemma 2.3). In this paper we will consider the consequence when \( A \) is a \( k \)-idempotent matrix.

When \( A \) is a \( k \)-idempotent matrix, we can easily proof \( A^D = A^{k-2} \). Then, we can change notations (3.1) and (3.2) to be

\[
K = A^{k-2}C, \quad H = BA^{k-2}, \quad \Gamma = HK, \quad Z = I - BA^{k-2}C, \quad (3.8)
\]
\[
P = (I - A^{k-1})C, \quad Q = B(I - A^{k-1}). \quad (3.9)
\]

**Theorem 3.3** Let \( A \) be a \( k \)-idempotent matrix, suppose \( AC = C \), \( \text{ind}(Z) = k \), then

\[
(A - CB)^D = A^{k-2} + KZ^DH - K(Z^D)^2Q - KZ\sum_{i=0}^{k-1}Z^iH. \quad (3.10)
\]

**Proof** The proof is similar to Theorem 3.1.

**Corollary 3.4** Let \( A \) be a \( k \)-idempotent matrix, suppose \( BA = B \), \( \text{ind}(Z) = k \), then

\[
(A - CB)^D = A^{k-2} + KZ^DH - P(Z^D)^2H - \sum_{i=0}^{k-1}KZ^iZ\pi H. \quad (3.11)
\]

**Proof** The proof is similar to Theorem 3.1.

**Theorem 3.5** Let \( A, B, C \) be diagonalizable. Suppose \( A, B, C \) commute,

\[
\text{rank}(A) = \text{rank}(B) = \text{rank}(C)
\]
and \( \sigma(A) \cap \sigma(BC) = \emptyset \), then

\[
(A - CB)^D = A^D + A^DC(I - BA^D C)^D BA^D, \quad (3.12)
\]

where \( \sigma(A) \) is the eigenvalues of \( A \).

**Proof** Because \( A, B \) and \( C \) are diagonalizable and they can commute, there is a nonsingular matrix \( S \) such that \( S^{-1}AS, S^{-1}BS, S^{-1}CS \) are diagonal. We denote

\[
S^{-1}AS = \begin{bmatrix}
\Lambda_1 & 0 \\
0 & 0
\end{bmatrix}, \quad S^{-1}BS = \begin{bmatrix}
\Lambda_2 & 0 \\
0 & 0
\end{bmatrix}, \quad S^{-1}CS = \begin{bmatrix}
\Lambda_3 & 0 \\
0 & 0
\end{bmatrix},
\]

where each of matrices \( \Lambda_1, \Lambda_2 \) and \( \Lambda_3 \) is full rank diagonal and its diagonal line elements are eigenvalues. Their rank is equal to \( A \). Then

\[
(A - CB)^D = S \begin{bmatrix}
(\Lambda_1 - \Lambda_3 \Lambda_2)^{-1} & 0 \\
0 & 0
\end{bmatrix} S^{-1},
\]
as \( \sigma(A) \cap \sigma(BC) = \emptyset \), \( \Lambda_1 - \Lambda_3 \Lambda_2 \) is nonsingular. At the same time, we have

\[
A^D + A^DC(I - BA^DC)^D BA^D = S \begin{bmatrix}
\Lambda_1^{-1} + \Lambda_1^{-1} \Lambda_3(I - \Lambda_2 \Lambda_1^{-1} \Lambda_3)^{-1} \Lambda_2 \Lambda_1^{-1} & 0 \\
0 & 0
\end{bmatrix} S^{-1}
\]
and

\[
(\Lambda_1 - \Lambda_3 \Lambda_2)^{-1} = \Lambda_1^{-1} + \Lambda_1^{-1} \Lambda_3(I - \Lambda_2 \Lambda_1^{-1} \Lambda_3)^{-1} \Lambda_2 \Lambda_1^{-1}, \quad (3.13)
\]
so we have \( (A - CB)^D = A^D + A^DC(I - BA^D C)^D BA^D \).
References


修正矩阵$A - CB$的Drazin逆

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摘要：本文研究了修正矩阵Drazin逆的表示形式，利用$k$次方等矩阵和对角化矩阵的性质，减弱了文献[4]中的条件，获得了新的Drazin逆的表示形式。

关键词：修正矩阵; Drazin 逆; Schur 补

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