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THE DRAZIN INVERSE OF A MODIFIED MATRIX $A-CB$

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Abstract: In this paper, we study the representations of the Drazin inverse of a modified matrix $A-CB$. By the properties of the k-idempotent matrix and the diagonalizable matrix, we get some new representations of the Drazin inverse through weakened conditions of literature [4].

Keywords: modified matrix; Drazin inverse; Schur complement

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1 Introduction

The problem of the Drazin inverse was discussed widely in [1–12]. The Drazin inverse was used to be applied in sigular differential difference equations, Markov chains and numerical analysis in [1–3]. The Drazin inverse of the modified matrices was studied by many people [4–6], as a modified matrix can be seen as the sum of two matrices or a matrix added a perturbed element. In [4], Wei Yiming gave the expression for the Drazin inverse of $A-CB$; Liu Xifu weakened the condition of [4] and gave another expression in [5]; the Drazin inverse of $A-CD^DB$ was given in [6]. In this paper, we weaken the conditions of [4–5] and give different results.

2 Definitions and Basic Results

Definition 2.1 Let $\mathbb{C}^{n\times n}$ denote the set of $n\times n$ complex matrices. The Drazin inverse of $A\in\mathbb{C}^{n\times n}$ is the unique matrix A^D satisfying the relations:

$$A^DAA^D=A^D,\quad A^DA=AA^D,\quad A^{k+1}A^D=A^k, \tag{2.1}$$

where k is the smallest non-negative integer such that rank(A^k) =rank(A^{k+1}), i.e., k =ind(A), the index of A. The case when ind$(A)=1$, the Drazin inverse is called the group inverse of A and it is denoted by $A^{\#}$. We denote by A^{π} corresponding to the eigenvalue 0 that is given by $A^{\pi}=I-AA^D$.

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Lemma 2.2 [4] Suppose $P=0$, $Q=0$ and $C(I-ZZ^D)B=0$. Then

$$(A-CB)^D = A^D + KZ^DH, \tag{2.2}$$

where denote $K=A^DC$, $H=BA^D$, $Z=I-BA^DC$, $P=(I-AA^D)C$ and $Q=B(I-A^DA)$.

Lemma 2.3 [5] Let A be an idempotent matrix, suppose $P=0$, $\mathrm{ind}(Z)=k$, then

$$(A-CB)^D = A + CZ^DH - C(Z^D)^2Q - CZ^\pi \sum_{i=0}^{k-1} Z^iH, \tag{2.3}$$

where denote $K=AC$, $H=BA$, $Z=I-BAC$, $P=(I-A)C$ and $Q=B(I-A)$, especially, $Z=I-BC$ at here.

Lemma 2.4 [6] Let A, B, C and D be complex matrices, where $\mathrm{ind}(A)=k$. If $A^\pi C=0$, $CZ^\pi=0$, $Z^\pi B=0$, $CD^\pi=0$ and $D^\pi B=0$, then

$$(A-CD^DB)^D = A^D + A^DCZ^DBA^D - \sum_{i=0}^{k-1}(A^D+A^DCZ^DBA^D)^{i+1}A^DCZ^DBA^iA^\pi, \tag{2.4}$$

where denote the schur complement $Z=D-BA^DC$, furthermore, $\mathrm{ind}(A-CD^DB)\le \mathrm{ind}(A)$.

3 Main Theorems and Proofs

First, we definite some notation similar to the reference [4]. Let

$$K=A^DC, \quad H=BA^D, \quad \Gamma=HK, \quad Z=I-BA^DC \tag{3.1}$$

and

$$P=(I-AA^D)C, \quad Q=B(I-A^DA). \tag{3.2}$$

Theorem 3.1 Let A, B and C be complex matrices, where $\mathrm{ind}(A)=k$. If $A^\pi C=0$ and $CZ^\pi B=0$, then

$$(A-CB)^D = A^D + KZ^DH - \sum_{i=0}^{k-1}(A^D+KZ^DH)^{i+1}KZ^DBA^iA^\pi. \tag{3.3}$$

furthermore, $\mathrm{ind}(A-CB)\le \mathrm{ind}(A)$.

Proof Let $X = A^D + KZ^DH - \sum\limits_{i=0}^{k-1}(A^D+KZ^DH)^{i+1}KZ^DBA^iA^\pi$. Then,

$$\begin{aligned}
(A-CB)X &= (A-CB)[A^D + KZ^DH - \sum_{i=0}^{k-1}(A^D+KZ^DH)^{i+1}KZ^DBA^iA^\pi] \\
&= (A-CB)(A^D+KZ^DH) - (A-CB)(A^D+KZ^DH)\sum_{i=0}^{k-1}(A^D+KZ^DH)^iKZ^DBA^iA^\pi \\
&= AA^D - AA^D\sum_{i=0}^{k-1}(A^D+KZ^DH)^iKZ^DBA^iA^\pi \\
&= AA^D - \sum_{i=0}^{k-1}(A^D+KZ^DH)^iKZ^DBA^iA^\pi.
\end{aligned} \tag{3.4}$$

At the same time, we get

$$\begin{aligned}
X(A-CB) &= [A^D+KZ^DH-\sum_{i=0}^{k-1}(A^D+KZ^DH)^{i+1}KZ^DBA^iA^\pi](A-CB)\\
&= (A^D+KZ^DH)(A-CB)\\
&\quad -\sum_{i=0}^{k-1}(A^D+KZ^DH)^{i+1}KZ^DBA^iA^\pi(A-CB)\\
&= AA^D-A^DCZ^DBA^\pi-\sum_{i=0}^{k-1}(A^D+KZ^DH)^{i+1}KZ^DBA^{i+1}A^\pi\\
&= AA^D-\sum_{i=0}^{k-1}(A^D+KZ^DH)^iKZ^DBA^iA^\pi. \qquad (3.5)
\end{aligned}$$

From (3.4) and (3.5) it follows that $(A-CB)X=X(A-CB)$.

Now, using (3.5) and $A^\pi X=0$, we obtain

$$(X(A-CB)-I)X=0,$$

i.e., $X(A-CB)X=X$.

Finally, we will prove that $(A-CB)-(A-CB)^2X$ is a nilpotent matrix. Using $A^\pi C=0, CZ^\pi B=0$, and expressions (3.4) conveniently, it can be proved that

$$(A-CB)-(A-CB)^2X=AA^\pi+\sum_{i=0}^{k-1}(A^D+KZ^DH)^iKZ^DBA^{i+1}A^\pi.$$

by induction on integer $j\geq 1$, we have

$$[(A-CB)-(A-CB)^2X]^j=A^jA^\pi+\sum_{i=0}^{k-1}(A^D+KZ^DH)^iKZ^DBA^{i+j}A^\pi.$$

Then we get

$$[(A-CB)-(A-CB)^2X]^k=0. \qquad (3.6)$$

where $k=\mathrm{ind}(A)$. Therefore, we get $(A-CB)^{k+1}X=(A-CB)^k$ and $\mathrm{ind}(A-CB)\leq\mathrm{ind}(A)$. The theorem is proved completely.

Corollary 3.2 Let A, B and C be complex matrices, where $\mathrm{ind}(A)=k$. If $BA^\pi=0$ and $CZ^\pi B=0$, then

$$(A-CB)^D=A^D+KZ^DH-\sum_{i=0}^{k-1}KZ^DBA^iA^\pi(A^D+KZ^DH)^{i+1}, \qquad (3.7)$$

furthermore, $\mathrm{ind}(A-CB)\leq\mathrm{ind}(A)$.

Proof The proof is similar to Theorem 3.1.

In the reference [5], it was discussed the Drazin inverse of a modified matrix $A-CB$, where A is an idempotent matrix (Lemma 2.3). In this paper we will consider the consequence when A is a k-idempotent matrix.

When A is a k-idempotent matrix, we can easily proof $A^D = A^{k-2}$. Then, we can change notations (3.1) and (3.2) to be

$$K = A^{k-2}C, \quad H = BA^{k-2}, \quad \Gamma = HK, \quad Z = I - BA^{k-2}C, \tag{3.8}$$

$$P = (I - A^{k-1})C, \quad Q = B(I - A^{k-1}). \tag{3.9}$$

Theorem 3.3 Let A be a k-idempotent matrix, suppose $AC = C$, $\mathrm{ind}(Z) = k$, then

$$(A-CB)^D = A^{k-2} + KZ^DH - K(Z^D)^2Q - KZ^{\pi}\sum_{i=0}^{k-1} Z^iH. \tag{3.10}$$

Proof The proof is similar to Theorem 3.1.

Corollary 3.4 Let A be a k-idempotent matrix, suppose $BA = B$, $\mathrm{ind}(Z) = k$, then

$$(A-CB)^D = A^{k-2} + KZ^DH - P(Z^D)^2H - \sum_{i=0}^{k-1} KZ^iZ^{\pi}H. \tag{3.11}$$

Proof The proof is similar to Theorem 3.1.

Theorem 3.5 Let A, B, C be diagonalizable. Suppose A, B, C commute,

$$\mathrm{rank}(A) = \mathrm{rank}(B) = \mathrm{rank}(C)$$

and $\sigma(A) \cap \sigma(BC) = \varnothing$, then

$$(A-CB)^D = A^D + A^DC(I - BA^DC)^DBA^D, \tag{3.12}$$

where $\sigma(A)$ is the eigenvalues of A.

Proof Because A, B and C are diagonalizable and they can commute, there is a nonsingular matrix S such that $S^{-1}AS$, $S^{-1}BS$, $S^{-1}CS$ are diagonal. We denote

$$S^{-1}AS = \begin{bmatrix} \Lambda_1 & o \\ o & o \end{bmatrix} \quad S^{-1}BS = \begin{bmatrix} \Lambda_2 & o \\ o & o \end{bmatrix} \quad S^{-1}CS = \begin{bmatrix} \Lambda_3 & o \\ o & o \end{bmatrix},$$

where each of matrices Λ_1, Λ_2 and Λ_3 is full rank diagonal and its diagonal line elements are eigenvalues. Their rank is equal to A. Then

$$(A-CB)^D = S\begin{bmatrix} (\Lambda_1 - \Lambda_3\Lambda_2)^{-1} & o \\ o & o \end{bmatrix} S^{-1},$$

as $\sigma(A) \cap \sigma(BC) = \varnothing$, $\Lambda_1 - \Lambda_3\Lambda_2$ is nonsingular. At he same time, we have

$$A^D + A^DC(I - BA^DC)^DBA^D = S\begin{bmatrix} \Lambda_1^{-1} + \Lambda_1^{-1}\Lambda_3(I - \Lambda_2\Lambda_1^{-1}\Lambda_3)^{-1}\Lambda_2\Lambda_1^{-1} & o \\ o & o \end{bmatrix} S^{-1}$$

and

$$(\Lambda_1 - \Lambda_3\Lambda_2)^{-1} = \Lambda_1^{-1} + \Lambda_1^{-1}\Lambda_3(I - \Lambda_2\Lambda_1^{-1}\Lambda_3)^{-1}\Lambda_2\Lambda_1^{-1}, \tag{3.13}$$

so we have $(A-CB)^D = A^D + A^DC(I - BA^DC)^DBA^D$.

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修正矩阵$A - CB$的Drazin逆

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摘要: 本文研究了修正矩阵Drazin逆的表示形式. 利用k次幂等矩阵和可对角化矩阵的性质, 减弱了文献[4]中的条件, 获得了新的Drazin逆的表示形式.

关键词: 修正矩阵; Drazin 逆; Schur 补

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