COMPARISON THEOREM FOR SOLUTIONS OF BSDES DRIVEN BY CONTINUOUS SEMI-MARTINGALES

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Abstract: Comparison theorem for solutions of one-dimensional backward stochastic equation (BSDE for short) was first established by Peng [1]. In this paper, we study the BSDEs driven by continuous semi-martingale satisfying Lipschitz condition. We generalize the comparison theorem to this case and prove it by using techniques which are different from those of Peng [1]. Our method is more direct and simpler.

Keywords: backward stochastic differential equations; comparison theorem; continuous semi-martingale

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1 Introduction

The backward stochastic differential equations (BSDEs) theory was a focus of great interest in recent years. To solve a classical BSDE, we look for a couple of processes \((y, z)\) which satisfies the equation

\[
y_t = \xi + \int_t^T f(s, y_s, z_s) ds - \int_t^T z_s dB_s, \quad 0 \leq t \leq T,
\]

where \(T > 0\) is a finite constant termed the time horizon, \(\xi\) is a one-dimensional random variable termed the terminal condition, the random function \(f : \Omega \times [0, T] \times R \times R \rightarrow R\) is progressively measurable for each \((y, z)\) termed the generator of the BSDE (1), and \(B\) is a d-dimensional Brownian motion. The solution \((y, z)\) is a pair of adapted processes. The triple \((\xi, T, f)\) is called the coefficients (parameters) of the BSDE (1).

The classical BSDE theory is taken the Brownian motion as the noise source, but the Brown motion is one kind of extreme idealized model, which causes the classical BSDE theory to receive certain limit in the application. Therefore, many scholars try to study the BSDE driven by other noise. For example, the corresponding work can be referenced in Situ (1997), Wang (1999), Nualart and Schoutens (2001), Bahlali (2003), Li (2005) and...
Ren and Hu (2007). In particular, existence and uniqueness results of the solutions of BSDE with continuous semi-martingale under Lipschitz condition were obtained by Wang (1999).

The comparison theorems, which are an important and effective technique in the theory of SDE, were established by Peng [1] and Cao-Yan [3]. In this paper, we generalize their comparison theorems to the case where the BSDEs were driven by continuous semi-martingale.

2 Preliminaries and Lemmas

Let us first introduce some assumptions and notations, which we will use in this paper. For what follows, let us fix a number $0 < T < +\infty$. Let $(W_t)_{t \leq T}$ be the standard $d$-dimensional Brownian motion defined on the canonical space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \leq T}, P)$, where $T < +\infty$ is a fixed time. We shall denote by $\mathcal{P}$, the predictable $\sigma$-field. Let $M^2_c$ be the continuous square-integrable martingale space. Let $S^2$ denote the set of $\mathcal{F}_t$-adapted càdlàg $\mathbb{R}^m$-valued process $\{X_t, 0 \leq t \leq T\}$ with the property

$$\|X\|_{S^2} = \left( E\left[ \sup_{0 \leq t \leq T} \|X_t\|^2 \right] \right)^{1/2} < +\infty.$$ 

Let $L^2(W)$ be the set of $\mathcal{F}_t$-predictable $\mathbb{R}^d$-valued processes $\{Z_t, 0 \leq t \leq T\}$ which satisfy

$$\|Z\|_{L^2(W)} = \left( E\left[ \int_0^T |Z_t|^2 d\langle M \rangle_t \right] \right)^{1/2} < +\infty.$$ 

In this paper, we consider the following one-dimensional backward stochastic differential equation

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) d\langle M \rangle_s + A_T - A_t - \int_t^T Z_s dM_s, \quad 0 \leq t \leq T,$$

where $t \in [0, T]$, $\xi$ an $\mathcal{F}_T$-measurable and square-integrable random variable, $M \in M^2_c$, $M_0 = 0$, $(A_t)_{0 \leq t \leq T}$ an $\mathcal{F}_t$-adapted càdlàg process and $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{d+1})$ measurable function, which satisfies

(H.1) $f(\cdot, 0, 0) \in L^2(W)$;

(H.2) Lipschitz conditions: there exist $C > 0$ such that

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq C(|x_1 - x_2| + |y_1 - y_2|) \text{ a.s. } \forall t \in [0, T], \ x_1, x_2, y_1, y_2 \in \mathbb{R}.$$ 

For given $(\xi, f, A)$, the solution to BSDE (2) means a pair of process $(Y, Z)$ in $S^2 \otimes L^2(W)$ which satisfies (2). Under the above conditions on $f$, Wang [2] proved the existence and uniqueness of the solutions of BSDE (2). We shall generalize the comparison theorem to the case and prove it by using techniques which are different from those of Peng. Our method is more direct and simpler.

3 A Comparison Theorem

Our main result is the following:
Theorem 3.1  Let \((Y^i, Z^i) \in S^2 \times L^2(W), ~i = 1, 2\) be the unique solutions to the following equations:

\[
Y^1_t = \xi^1 + \int_0^t f^1(s, Y^1_s, Z^1_s)d\langle M \rangle_s + A^1_T - A^1_t - \int_t^T Z^1_s dM_s, 0 \leq t \leq T
\]  \hspace{1cm} (3)

and

\[
Y^2_t = \xi^2 + \int_0^t f^2(s, Y^2_s, Z^2_s)d\langle M \rangle_s + A^2_T - A^2_t - \int_t^T Z^2_s dM_s, 0 \leq t \leq T,
\]  \hspace{1cm} (4)

where \(\xi^i \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}), \ i = 1, 2\), \(f^1\) satisfies (H.1) and (H.2), \(f^2\) is a progressively measurable process such that \(E\int_0^T |f^2|^2 d\langle M \rangle_t < +\infty\), \(A^i \in S^2, A^i_0 = 0\).

(i) If \(\xi^1 \leq \xi^2\) a.s. \(f^1(s, Y^1, Z^1) \leq f^2(s, Y^2, Z^2)\) a.s. a.e.(s) and if \((A^i_t)(i = 1, 2)\) are continuous, and \(\{A^2_t - A^1_t\}\) is an increasing process, then \(\forall 0 \leq t \leq T\), we have \(Y^1_t \leq Y^2_t\) a.s.;

(ii) If \(\xi^1 \geq \xi^2\) a.s. \(f^1(s, Y^1, Z^1) \geq f^2(s, Y^2, Z^2)\) a.s. a.e.(s) and if \((A^i_t)(i = 1, 2)\) are continuous, and \(\{A^1_t - A^2_t\}\) is an increasing process, then \(\forall 0 \leq t \leq T\), we have \(Y^1_t \geq Y^2_t\), a.s.

In order to prove Theorem 3.1, we need the following lemma.

Lemma 3.2  Let \(X_t = X_0 + M_t + V_t\) be a continuous semi-martingale, where \((M_t)\) a continuous local martingale with \(M_0 = 0\) and \((V_t)\) a continuous process of finite variation with \(V_0 = 0\). Then

\[
X_t = X_0^2 + 2 \int_0^t X_s^+dM_s + 2 \int_0^t X_s^+dV_s + \int_0^t I_{(X_s > 0)}d\langle M \rangle_s.
\]

Proof  Applying Itô formula to the Tanaka-Meyer formula, it is easy to prove this. For details we refer to Cao [3].

Proof of Theorem 3.1  We only need to prove (i), because (ii) can be deduced from (i) easily. Let \(Y^1\) and \(Y^2\) be the solutions to (3) and (4), respectively, and denote \(Y_t = Y^1_t - Y^2_t\), \(\xi = \xi^1 - \xi^2\), \(Z_t = Z^1_t - Z^2_t\), \(A_t = A^1_t - A^2_t\).

From (3) and (4), we have

\[
Y_t = \xi + \int_t^T [f^1(s, Y^1_s, Z^1_s) - f^2(s, Y^2_s, Z^2_s)]d\langle M \rangle_s + A_T - A_t - \int_t^T Z_s dM_s
\]

\[= Y_0 - \int_0^t [f^1(s, Y^1_s, Z^1_s) - f^2(s, Y^2_s, Z^2_s)]d\langle M \rangle_s + A_t + \int_0^T Z_s dM_s.
\]

It is easily seen that \(Y_t\) is a continuous semi-martingale. By Lemma 3.2, we obtain

\[
Y_t^+ = \xi^+ + 2 \int_t^T Y_s^+ [f^1(s, Y^1_s, Z^1_s) - f^2(s, Y^2_s, Z^2_s)]d\langle M \rangle_s - 2 \int_t^T Y_s^+ Z_s dM_s
\]

\[+ 2 \int_t^T Y_s^+dA_s - \int_t^T I_{(Y_s > 0)}|Z_s|^2 d\langle M \rangle_s.
\]
Since \( \xi \leq 0 \), \( (A_t) \) is a decreasing process, we get
\[
Y_t^+ + \int_t^T I_{(Y_s > 0)}|Z_s|^2 d\langle M \rangle_s \\
\leq 2 \int_t^T Y_s^+ [f^1(s, Y_s^1, Z_s^1) - f^1(s, Y_s^2, Z_s^2)] d\langle M \rangle_s - 2 \int_t^T Y_s^+ Z_s dM_s \\
\leq 2 \int_t^T Y_s^+ [f^1(s, Y_s^1, Z_s^1) - f^1(s, Y_s^2, Z_s^2)] d\langle M \rangle_s - 2 \int_t^T Y_s^+ Z_s dM_s.
\]

Obviously, \( \{ \int_t^T Y_s^+ Z_s dM_s, \ t \in [0, T] \} \) is a martingale (cf. [3]). Then by (H.2) and the elementary inequality \( 2|uv| \leq \frac{1}{c}u^2 + cv^2 \) for any \( c > 0 \), we have
\[
Ey_t^+ + 1 \int_t^T I_{(Y_s > 0)}|Z_s|^2 d\langle M \rangle_s \\
\leq 2E \int_t^T Y_s^+ [f^1(s, Y_s^1, Z_s^1) - f^1(s, Y_s^2, Z_s^2)] d\langle M \rangle_s \\
= 2E \int_t^T Y_s^+ I_{(Y_s > 0)}[f^1(s, Y_s^1, Z_s^1) - f^1(s, Y_s^2, Z_s^2)] d\langle M \rangle_s \\
\leq E \int_t^T [cY_s^+ + \frac{1}{c}I_{(Y_s > 0)}(f^1(s, Y_s^1, Z_s^1) - f^1(s, Y_s^2, Z_s^2))]^2 d\langle M \rangle_s \\
\leq E \int_t^T [cY_s^+ + \frac{2K}{c}I_{(Y_s > 0)}(Y_s^2 + Z_s^2)] d\langle M \rangle_s \\
= E \int_t^T [(c + \frac{2K}{c})Y_s^+ + \frac{2K}{c}I_{(Y_s > 0)}|Z_s|^2] d\langle M \rangle_s.
\]

Choosing \( c = 2K \), we obtain
\[
Ey_t^+ \leq (2K + 1) \int_t^T Ey_s^+ d\langle M \rangle_s,
\]
which implies that \( Ey_t^+ = 0 \) for all \( t \in [0, T] \) by Gronwall’s lemma. As \( Y_t^+ \) is continuous, we have \( Y_t^+ = 0 \), for \( t \in [0, T] \) a.s. The theorem is proved.

References


由连续半鞅驱动的倒向随机微分方程解的比较定理

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