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COMPARISON THEOREM FOR SOLUTIONS OF BSDES DRIVEN BY CONTINUOUS SEMI-MARTINGALES

LI Shi-yu, LI Wen-xue, GAO Wu-jun

(*Faculty of Science, Jiangxi University of Science and Technology, Ganzhou 341000, China*)

Abstract: Comparison theorem for solutions of one-dimensional backward stochastic equation (BSDE for short) was first established by Peng [1]. In this paper, we study the BSDEs driven by continuous semi-martingale satisfying Lipschitz condition. We generalize the comparison theorem to this case and prove it by using techniques which are different from those of Peng [1]. Our method is more direct and simpler.

Keywords: backward stochastic differential equations; comparison theorem; continuous semi-martingale

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1 Introduction

The backward stochastic differential equations (BSDEs) theory was a focus of great interest in recent years. To solve a classical BSDE, we look for a couple of processes (y, z) which satisfies the equation

$$y_t = \xi + \int_t^T f(s, y_s, z_s)ds - \int_t^T z_s dB_s,\ 0 \le t \le T, \tag{1}$$

where $T > 0$ is a finite constant termed the time horizon, ξ is a one-dimensional random variable termed the terminal condition, the random function $f : \Omega \times [0, T] \times R \times R \to R$ is progressively measurable for each (y, z) termed the generator of the BSDE (1), and B is a d-dimensional Brownian motion. The solution (y, z) is a pair of adapted processes. The triple (ξ, T, f) is called the coefficients (parameters) of the BSDE (1).

The classical BSDE theory is taken the Brownian motion as the noise source, but the Brown motion is one kind of extreme idealized model, which causes the classical BSDE theory to receive certain limit in the application. Therefore, many scholars try to study the BSDE driven by other noise. For example, the corresponding work can be referenced in Situ (1997), Wang (1999), Nualart and Schoutens (2001), Bahlali (2003), Li (2005) and

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Biography: Li Shiyu(1983–), male, born at Ganzhou, Jiangxi, lecturer, major in backward stochastic differential equation. E–mail:lishiyu83@163.com.

Ren and Hu (2007). In particular, existence and uniqueness results of the solutions of BSDE with continuous semi-martingle under Lipschitz condition were obtained by Wang (1999).

The comparison theorems, which are an important and effective technique in the theory of SDE, were established by Peng [1] and Cao-Yan [3]. In this paper, we generalize their comparison theorems to the case where the BSDEs were driven by continuous semi-martingale.

2 Preliminaries and Lemmas

Let us first introduce some assumptions and notations, which we will use in this paper. For what follows, let us fix a number $0 < T < +\infty$. Let $(W_t)_{(t\leq T)}$ be the standard d-dimensional Brownian motion defined on the canonical space $(\Omega, \mathscr{F}_T, (\mathscr{F}_t)_{(t\leq T)}, P)$, where $T < +\infty$ is a fixed time.We shall denote by $\mathcal{P}$, the predictable σ-field. Let M_c^2 be the continuous square-integrable martingale space. Let S^2 denote the set of $\mathscr{F}_t$-adapted càdlàg R^m-valued process $\{X_t, 0 \leq t \leq T\}$ with the property

$$\|X\|_{S^2} = (E[\sup_{0\leq t\leq T} \|X_t\|^2)^{1/2} < +\infty.$$

Let $L^2(W)$ be the set of $\mathscr{F}_t$-predictable R^d-valued processes $\{Z_t, 0 \leq t \leq T\}$ which satisfy

$$\|Z\|_{L^2(W)} = (E\int_0^T |Z_t|^2 d\langle M\rangle_t)^{1/2} < +\infty.$$

In this paper, we consider the following one-dimensional backward stochastic differential equation

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) d\langle M\rangle_s + A_T - A_t - \int_t^T Z_s dM_s,\ 0 \leq t \leq T, \tag{2}$$

where $t \in [0, T]$, ξ an $\mathscr{F}_T$-measurable and square-integrable random variable, $M \in M_c^2, M_0 = 0$, $(A_t)_{(0\leq t\leq T)}$ an $\mathscr{F}_t$-adapted càdlàg process and $f : \Omega \times [0, T] \times R \times R^d \to R$ a $\mathcal{P} \bigotimes \mathscr{B}(R^{d+1})$ measurable function, which satisfies

(H.1) $f(\bullet, 0, 0) \in L^2(W)$;

(H.2) Lipschitz conditions: there exist $C > 0$ such that

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq C(|x_1 - x_2| + |y_1 - y_2|) \text{ a.s. } \forall t \in [0, T],\ x_1, x_2, y_1, y_2 \in R.$$

For given (ξ, f, A), the solution to BSDE (2) means a pair of process (Y, Z) in $S^2 \bigotimes L^2(W)$ which satisfies (2). Under the above conditions on f, Wang [2] proved the existence and uniqueness of the solutions of BSDE (2). We shall generalize the comparison theorem to the case and prove it by using techniques which are different from those of Peng. Our method is more direct and simpler.

3 A Comparison Theorem

Our main result is the following:

Theorem 3.1 Let $(Y^i, Z^i) \in S^2 \times L^2(W),\ i = 1, 2$ be the unique solutions to the following equations:

$$Y_t^1 = \xi^1 + \int_t^T f^1(s, Y_s^1, Z_s^1) d\langle M\rangle_s + A_T^1 - A_t^1 - \int_t^T Z_s^1 dM_s, 0 \le t \le T \tag{3}$$

and

$$Y_t^2 = \xi^2 + \int_t^T f^2(s, Y_s^2, Z_s^2) d\langle M\rangle_s + A_T^2 - A_t^2 - \int_t^T Z_s^2 dM_s, 0 \le t \le T, \tag{4}$$

where $\xi^i \in L^2(\Omega, \mathscr{F}_T, P)$, $i = 1, 2$, f^1 satisfies (H.1) and (H.2), f^2 is a progressively measureable process such that $E\int_0^T |f_t^2|^2 d\langle M\rangle_t < +\infty, A^i \in S^2, A_0^i = 0$.

(i) If $\xi^1 \le \xi^2$ a.s. $f^1(s, Y^2, Z^2) \le f^2(s, Y^2, Z^2)$ a.s. a.e.(s) and if $(A_t^i)(i = 1, 2)$ are continuous, and $\{A_t^2 - A_t^1\}$ is an increasing process, then $\forall 0 \le t \le T$, we have $Y_t^1 \le Y_t^2$ a.s.;

(ii) If $\xi^1 \ge \xi^2$ a.s. $f^1(s, Y^2, Z^2) \ge f^2(s, Y^2, Z^2)$ a.s. a.e.(s) and if (A_t^i) $(i = 1, 2)$ are continuous, and $\{A_t^1 - A_t^2\}$ is an increasing process, then $\forall 0 \le t \le T$, we have $Y_t^1 \ge Y_t^2$, a.s..

In order to prove Theorem 3.1, we need the following lemma.

Lemma 3.2 Let $X_t = X_0 + M_t + V_t$ be a continuous semi-martingale, where (M_t) a continuous local martingale with $M_0 = 0$ and (V_t) a continuous process of finite variation with $V_0 = 0$. Then

$$X_t^{+^2} = X_0^{+^2} + 2\int_0^t X_s^+ dM_s + 2\int_0^t X_s^+ dV_s + \int_0^t I_{(X_s>0)} d\langle M\rangle_s.$$

Proof Applying Itö formula to the Tanaka-Meyer formula, it is easy to prove this. For details we refer to Cao [3].

Proof of Theorem 3.1 We only need to prove (i), because (ii) can be deduced from (i) easily. Let Y^1 and Y^2 be the solutions to (3) and (4), respectively, and denote $Y_t = Y_t^1 - Y_t^2$, $\xi = \xi^1 - \xi^2$, $Z_t = Z_t^1 - Z_t^2$, $A_t = A_t^1 - A_t^2$.

From (3) and (4), we have

$$\begin{aligned} Y_t &= \xi + \int_t^T [f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2)] d\langle M\rangle_s + A_T - A_t - \int_t^T Z_s dM_s \\ &= Y_0 - \int_0^t [f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2)] d\langle M\rangle_s - A_t + \int_0^t Z_s dM_s. \end{aligned}$$

It is easily seen that Y_t is a continuous semi-martingale. By Lemma 3.2, we obtain

$$\begin{aligned} Y_t^{+^2} =& \xi^{+^2} + 2\int_t^T Y_s^+ [f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2)] d\langle M\rangle_s - 2\int_t^T Y_s^+ Z_s dM_s \\ &+ 2\int_t^T Y_s^+ dA_s - \int_t^T I_{(Y_s>0)} |Z_s|^2 d\langle M\rangle_s. \end{aligned}$$

Since $\xi \leq 0$, (A_t) is a decreasing process, we get

$$\begin{aligned}
&Y_t^{+2} + \int_t^T I_{(Y_s>0)}|Z_s|^2 d\langle M\rangle_s \\
\leq& 2\int_t^T Y_s^+[f^1(s,Y_s^1,Z_s^1) - f^2(s,Y_s^2,Z_s^2)]d\langle M\rangle_s - 2\int_t^T Y_s^+ Z_s dM_s \\
\leq& 2\int_t^T Y_s^+[f^1(s,Y_s^1,Z_s^1) - f^1(s,Y_s^2,Z_s^2)]d\langle M\rangle_s - 2\int_t^T Y_s^+ Z_s dM_s.
\end{aligned}$$

Obviously, $\{\int_t^T Y_s^+ Z_s dM_s,\ t \in [0,T]\}$ is a martingale (cf. [3]). Then by (H.2) and the elementary inequality $2|uv| \leq \frac{1}{c}u^2 + cv^2$ for any $c > 0$, we have

$$\begin{aligned}
&EY_t^{+2} + E\int_t^T I_{(Y_s>0)}|Z_s|^2 d\langle M\rangle_s \\
\leq& 2E\int_t^T Y_s^+[f^1(s,Y_s^1,Z_s^1) - f^1(s,Y_s^2,Z_s^2)]d\langle M\rangle_s \\
=& 2E\int_t^T Y_s^+ I_{(Y_s>0)}[f^1(s,Y_s^1,Z_s^1) - f^1(s,Y_s^2,Z_s^2)]d\langle M\rangle_s \\
\leq& E\int_t^T [cY_s^{+2} + \frac{1}{c}I_{(Y_s>0)}(f^1(s,Y_s^1,Z_s^1) - f^1(s,Y_s^2,Z_s^2))^2]d\langle M\rangle_s \\
\leq& E\int_t^T [cY_s^{+2} + \frac{2K}{c}I_{(Y_s>0)}(Y_s^2 + Z_s^2)]d\langle M\rangle_s \\
=& E\int_t^T [(c + \frac{2K}{c})Y_s^{+2} + \frac{2K}{c}I_{(Y_s>0)}|Z_s|^2)]d\langle M\rangle_s.
\end{aligned}$$

Choosing $c = 2K$, we obtain

$$EY_t^{+2} \leq (2K+1)\int_t^T EY_s^{+2} d\langle M\rangle_s,$$

which implies that $EY_t^{+2} = 0$ for all $t \in [0,T]$ by Gronwall's lemma. As Y_t^{+2} is continuous, we have $Y_t^+ = 0$, for $t \in [0,T]$ a.s. The theorem is proved.

References

[1] Peng Shige. A generalized dynamic programming principle and Hamilton-Jacob-Bellman equation[J]. Stochastics, 1992, 38: 119–134.

[2] Wang Xiangjun. On backward stochastic differential equation by a continuous semi-martingale[J]. J. of Math., 1999, 19(1): 45–50.

[3] Cao Zhigang, Yan Jiaan. A comparison theorem for solutions of backward stochastic differential equations[J]. Advanced in Mathematics, 1999, 28: 304–308.

[4] Mao Xuerong. Adapted solutions of backward stochastic differential equations with non-Lipschitz coeffcients[J]. Stochastic Process Appl., 1995, 58: 281–292.

[5] Lepeltier J, Martin J. Backward stochastic differential equations with continuous coeffcient[J]. Statist. Probab. Lett., 1997, 32: 425–430.

[6] Li Shiyu, Gao Wujun. Solution of backward stochaistic differential equations driven by contious local martingales[J]. J. Jiangxi University of Science and Technology, 2009, 30(5): 71–73.

[7] Fan Shengjun, Jiang Long. Finite and infinite time interval BSDEs with non-Lipschitz coefficients[J]. Statistics and Probability Letters, 2010, 80: 962–968.

[8] Li Juan. Backward stochastic differential equations with general martingale[J]. J. Shandong University, 2005, 40(4): 70–76.

[9] Wang Y, Huang Z. Backward stochastic differential equations with non-Lipschitz coefficients[J]. Statist. Probab. Lett., 2009, 79: 1438–1443.

[10] Fan Shengjun, Jiang Long. One-dimensional BSDEs with finite and infinite time horizons[J]. Stochastic Proc. Appl., 2011, 121: 427–440.

[11] Pardoux E, Peng Shige. Adapted solution of a backward stochastic differential equation [J]. Systems Control Lett., 1990, 14: 55–66.

由连续半鞅驱动的倒向随机微分方程解的比较定理

李师煜 , 李文学 , 高武军

(江西理工大学理学院, 江西 赣州 341000)

摘 要: 彭实戈[1]首先建立了一维倒向随机微分方程的比较定理, 本文在Lipschitz条件下研究由连续半鞅驱动的倒向随机微分方程, 我们将比较定理推广到此类倒向随机微分方程, 并且证明方法比彭实戈[1]的更加直接和简单.

关 键 词: 倒向随机微分方程; 比较定理; 连续半鞅

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