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COMPOSITE IMPLICIT ITERATION PROCESS FOR A LIPSCHITZIAN PSEUDOCONTRACTIVE MAPPING

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Abstract: In this paper, the composite implicit iterative process for a Lipschitzian pseudocontractive mapping is studied. By using the corresponding equivalent inequality of pseudocontractive mapping, the sufficient and necessary conditions for the strong convergence of the composite implicit iterative process are obtained in Banach spaces, which generalize some related results.

Keywords: pseudocontractive mapping; fixed point; composite implicit iterative scheme; Banach space

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1 Introduction and Preliminaries

Throughout this work, we assume that E is a real Banach space, E^* is the dual space of E and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\|f\|, \|f\| = \|x\|\}, \quad \forall x \in E,$$

where $\langle \cdot, \cdot\rangle$ denotes duality pairing between E and E^*. A single-valued normalized duality mapping is denoted by j.

Let C be a nonempty subset of E, $T : C \to C$ be a mapping. We denote the set of fixed points of T by $F(T)$.

A mapping T with domain $D(T)$ and range $R(T)$ in E is called pseudocontractive [1], if there exists some $j(x - y) \in J(x - y)$ such that

$$\langle j(x - y), Tx - Ty\rangle \le \|x - y\|^2 \tag{1.1}$$

for all $x, y \in D(T)$. It is well known that [2] (1.1) is equivalent to the following:

$$\|x - y\| \le \|x - y + s[(I - T)x - (I - T)y]\| \tag{1.2}$$

for all $s > 0$ and all $x, y \in D(T)$.

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Theorem 1.1 [3] Let C be a convex compact subset of a Hilbert space H and $T : C \to C$ be a Lipschitzian pseudocontractive mapping. For any $x_1 \in C$, suppose the sequence $\{x_n\}$ is defined by

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \quad n \geq 1, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are two real sequences in $[0,1]$ satisfying

(i) $\alpha_n \leq \beta_n, n \geq 1$;

(ii) $\lim\limits_{n\to\infty} \beta_n = 0$;

(iii) $\Sigma_{n=1}^{\infty}\alpha_n\beta_n = \infty$.

Then $\{x_n\}$ converges strongly to a fixed point of T.

Remark 1.1 (1) Since $0 \leq \alpha_n \leq \beta_n \leq 1, n \geq 1$ and $\Sigma_{n=1}^{\infty}\alpha_n\beta_n = \infty$, the iterative sequence (1.3) couldn't be reduced to Mann iterative sequence by setting $\beta_n = 0$. The Mann iterative sequence [4] is defined by the following

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n \geq 1, \tag{1.4}$$

where $\{\alpha_n\}$ is a appropriate sequence in [0,1].

(2) Chidume and Mutangadura [5] gave an example to show that the Mann iterative sequence failed to be convergent to a fixed point point of Lipschitzian pseudocontractive mapping.

Let C be a nonempty convex subset of a real Banach space and $T : C \to C$ be a Lipschitzian pseudocontractive mapping, we introduce a composite implicit iteration process $\{x_n\}$ as follows:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_{n+1}, \quad n \geq 1, \end{cases} \tag{1.5}$$

where $\{\alpha_n\}$, $\{\beta_n\} \subset [0,1]$. When $\beta_n = 0$, $\forall n \geq 1$, (1.4) is the special form of (1.5).

In 1974, Deimling [6] proved the following fixed point theorem.

Theorem 1.2 Let E be a real Banach space, K a nonempty closed convex subset of E, and $T : K \to K$ a continuous strongly pseudocontractive mapping. Then, T has a unique fixed point in K.

Observe that if C is a nonempty closed convex subset of E and $T : C \to C$ is a Lipschitz pseudocontractive mapping, then for every $u \in C$ and $t, s \in (0,1)$, the mapping $S_{t,s} : C \to C$ is defined by $S_{t,s}x = (1-t)u + tT((1-s)u + sTx)$ satisfies that

$$\langle S_{t,s}x - S_{t,s}y, j(x-y)\rangle \leq tsL^2\|x-y\|^2$$

for all $x, y \in C$. Thus $S_{t,s}$ is strongly pseudocontractive if $tsL^2 < 1$. Since $S_{t,s}$ is also Lipschitiz, it follows from Theorem 1.2 that there exists a unique fixed point $x_{t,s} \in C$ of $S_{t,s}$ such that

$$x_{t,s} = (1-t)u + tT((1-s)u + sTx_{t,s}).$$

This shows that the implicit iteration sequence (1.5) can be employed for the approximation of fixed points of a Lipschitz pseudocontractive mapping.

In this paper, we give necessary and sufficient conditions for the strong convergence of iterative sequence (1.5) and Mann iterative sequence to a fixed point of a Lipschitzian pseudocontractive mapping in Banach spaces.

In order to prove main results, the following lemma is needed.

Lemma 1.1 [7] Let $\{a_n\}, \{b_n\}, \{c_n\}$ be sequences of nonnegative real numbers satisfying the inequality

$$a_{n+1} \leq (1 + c_n)a_n + b_n, \quad n \geq 1.$$

If $\Sigma_{n=1}^{\infty} c_n < +\infty$, $\Sigma_{n=1}^{\infty} b_n < +\infty$, then $\lim\limits_{n\to\infty} a_n$ exists.

2 Main Results

First, we prove the following lemma.

Lemma 2.1 Let E be a real Banach space and C a nonempty closed convex subset of E. Let $T : C \to C$ be a Lipschitzian pseudocontractive mapping with Lipschitz constant $L > 1$ and $F(T) \neq \emptyset$. Suppose that the sequence $\{x_n\}$ is defined by (1.5) such that $\sum\limits_{n=1}^{\infty} \alpha_n\beta_n < \infty$, $\alpha_n\beta_n L^2 < 1$ for all $n \geq 1$ and $\sum\limits_{n=1}^{\infty} \alpha_n^2 < \infty$. Then

(i) There exist a sequence $\{r_n\} \subseteq (0, \infty)$ and some positive integer, such that $\sum\limits_{n=1}^{\infty} r_n < \infty$ and

$$\|x_{n+1} - p\| \leq (1 + r_n)\|x_n - p\|$$

for all $p \in F(T)$ and $n \geq n_0$.

(ii) There exists a constant $M > 1$, for all integer $m \geq 1$ such that

$$\|x_{n+m} - p\| \leq M\|x_n - p\|$$

for all $p \in F(T)$.

Proof Let $p \in F(T)$. By (1.5), we have

$$\begin{aligned} x_n &= x_{n+1} + \alpha_n x_n - \alpha_n T y_n \\ &= x_{n+1} + \alpha_n(I - T)x_{n+1} + \alpha_n^2(x_n - Ty_n) + \alpha_n(Tx_{n+1} - Ty_n). \end{aligned} \tag{2.1}$$

Observe that

$$p = p + \alpha_n(I - T)p. \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$\begin{aligned} x_n - p &= x_{n+1} - p + \alpha_n[(I - T)x_{n+1} - (I - T)p] \\ &\quad + \alpha_n^2(x_n - Ty_n) + \alpha_n(Tx_{n+1} - Ty_n). \end{aligned} \tag{2.3}$$

Together with (2.3) and (1.2), we have

$$\begin{aligned}\|x_n-p\| &\geq \|x_{n+1}-p+\alpha_n[(I-T)x_{n+1}-(I-T)p]\| \\ &\quad -\alpha_n^2\|x_n-Ty_n\|-\alpha_n\|Tx_{n+1}-Ty_n\| \\ &\geq \|x_{n+1}-p\|-\alpha_n^2\|x_n-Ty_n\|-\alpha_n\|Tx_{n+1}-Ty_n\|.\end{aligned}$$

This implies that

$$\|x_{n+1}-p\| \leq \|x_n-p\|+\alpha_n^2\|x_n-Ty_n\|+\alpha_n\|Tx_{n+1}-Ty_n\|. \tag{2.4}$$

Next, we make the following estimations:

$$\begin{aligned}&\|y_n-p\| \leq (1-\beta_n)\|x_n-p\|+\beta_n\|Tx_{n+1}-p\| \\ \leq\ &(1-\beta_n)\|x_n-p\|+\beta_nL\|x_{n+1}-p\|, \qquad (2.5)\\ &\|x_n-Ty_n\| \leq \|x_n-p\|+\|p-Ty_n\| \\ \leq\ &\|x_n-p\|+L\|y_n-p\| \qquad (2.6)\end{aligned}$$

and

$$\begin{aligned}&\|Tx_{n+1}-Ty_n\| \leq L\|x_{n+1}-y_n\| = L\|x_n-y_n+\alpha_n(Ty_n-x_n)\| \\ \leq\ &L\beta_n\|Tx_{n+1}-x_n\|+L\alpha_n\|Ty_n-x_n\| \\ \leq\ &L^2\beta_n\|x_{n+1}-p\|+L\beta_n\|x_n-p\|+L\alpha_n(\|x_n-p\|+L\|y_n-p\|). \qquad (2.7)\end{aligned}$$

Substituting (2.5), (2.6) and (2.7) into (2.4) yields that

$$\begin{aligned}&\|x_{n+1}-p\| \leq \|x_n-p\|+\alpha_n^2[\|x_n-p\|+L(1-\beta_n)\|x_n-p\|+L^2\beta_n\|x_{n+1}-p\|] \\ &+\alpha_n[L^2\beta_n\|x_{n+1}-p\|+L\beta_n\|x_n-p\|+L\alpha_n(\|x_n-p\| \\ &+L(1-\beta_n)\|x_n-p\|+L^2\beta_n\|x_{n+1}-p\|)].\end{aligned}$$

This implies that

$$\begin{aligned}&(1-L^2\alpha_n^2\beta_n-L^2\alpha_n\beta_n-L^3\alpha_n^2\beta_n)\|x_{n+1}-p\| \\ \leq\ &(1+\alpha_n^2+2L\alpha_n^2+L\alpha_n\beta_n+L^2\alpha_n^2)\|x_n-p\|.\end{aligned}$$

Since $\lim\limits_{n\to\infty}\alpha_n\beta_n=0$, there exists integer $n_0>0$ such that $\alpha_n\beta_n \leq \frac{1}{6L^3}$ for all $n \geq n_0$ and

$$\begin{aligned}&1-L^2\alpha_n^2\beta_n-L^2\alpha_n\beta_n-L^3\alpha_n^2\beta_n \geq 1-L^2\cdot\frac{1}{6L^3}-L^2\cdot\frac{1}{6L^3}-L^3\cdot\frac{1}{6L^3} \\ =\ &\frac{5L-2}{6L} \geq \frac{5L-2L}{6L}=\frac{1}{2}.\end{aligned}$$

Therefore, we have

$$\|x_{n+1}-p\| \leq (1+r_n)\|x_n-p\|,$$

where $r_n = 2[(L^3+2L^2+L)\alpha_n\beta_n+(1+L)^2\alpha_n^2]$. Since $\sum\limits_{n=1}^{\infty}\alpha_n\beta_n<\infty$ and $\sum\limits_{n=1}^{\infty}\alpha_n^2<\infty$. Then $\sum\limits_{n=1}^{\infty} r_n<\infty$. By Lemma 1.1, we obtain that $\lim\limits_{n\to\infty}\|x_n-p\|$ exists. This completes the proof of part (i).

(ii) For any $m\geq 1$ and $n\geq n_0$, we have

$$\begin{aligned} & \|x_{n+m}-p\| \leq (1+r_{n+m-1})\|x_{n+m-1}-p\| \leq e^{r_{n+m-1}}\|x_{n+m-1}-p\| \\ \leq\ & e^{r_{n+m-1}}e^{r_{n+m-2}}\|x_{n+m-2}-p\| \\ & \vdots \\ \leq\ & e^{\Sigma_{k=n}^{n+m-1} r_k}\|x_n-p\| \\ \leq\ & e^{\Sigma_{k=n}^{\infty} r_n}\|x_n-p\| = M\|x_n-p\|, \end{aligned}$$

where $M=e^{\Sigma_{k=n}^{\infty} r_n}$. This completes the proof.

Theorem 2.1 Let E be a real Banach space and C a nonempty closed convex subset of E. Let $T: C\to C$ be a Lipschitzian pseudocontractive mapping with Lipschitz constant $L>1$ and $F(T)\neq\emptyset$. Suppose that the sequence $\{x_n\}$ is defined by (1.5) such that $\sum\limits_{n=1}^{\infty}\alpha_n\beta_n<\infty$, $\alpha_n\beta_n L^2<1$ for all $n\geq 1$ and $\sum\limits_{n=1}^{\infty}\alpha_n^2<\infty$. Then $\{x_n\}$ converges strongly to a fixed point of T if and only if $\liminf\limits_{n\to\infty} d(x_n,F(T))=0$, where $d(x_n,F(T))=\inf\limits_{q\in F(T)}\|x_n-q\|$.

Proof The necessity of Theorems 2.1 is obvious. We just need to prove the sufficiency. By Lemma 2.1, we have

$$d(x_{n+1},F(T))\leq (1+r_n)d(x_n,F(T)).$$

By Lemma 1.1 and the condition $\liminf\limits_{n\to\infty} d(x_n,F(T))=0$, then $\lim\limits_{n\to\infty} d(x_n,F(T))=0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence. For any $\varepsilon>0$, there exists an integer $n_1>n_0>0$ such that

$$d(x_n,F(T))<\frac{\varepsilon}{4M}$$

for all $n\geq n_1$. In particular, there exists $p_1\in F(T)$ and a constant $n_2>n_1$ such that

$$\|x_{n_2}-p_1\|<\frac{\varepsilon}{2M}. \tag{2.8}$$

Using Lemma 2.1 (ii) and (2.8), for all $n_2>n_1$ and $m\geq 1$, we have

$$\begin{aligned} & \|x_{n+m}-x_n\| \leq \|x_{n+m}-p_1\|+\|p_1-x_n\| \\ \leq\ & 2M\|x_{n_2}-p_1\| < 2M\cdot\frac{\varepsilon}{2M}=\varepsilon. \end{aligned}$$

Hence, $\{x_n\}$ is a Cauchy sequence. Since C is closed subset of E, so $\{x_n\}$ converges strongly to $p_0\in C$. It follows from $F(T)$ is a closed set and $\lim\limits_{n\to\infty} d(x_n,F(T))=0$ that $p_0\in F(T)$. This shows that $\{x_n\}$ converges strongly to a fixed point of T in C. This completes the proof.

Remark 2.1 Let $\beta_n = 0$ in iterative sequence (1.5), we can obtain strong convergence theorem of Mann iterative sequence from Theorem 2.1.

Theorem 2.2 Let E be a real Banach space and C a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N : C \to C$ be N Lipschitzian pseudocontractive mappings with Lipschitz constant $L_i > 1$ and $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Suppose that the sequence $\{x_n\}$ is defined as follows:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T_n x_{n+1}, \quad n \geq 1, \end{cases}$$

where $T_n = T_{n \bmod N}$, $\{\alpha_n\}$, $\{\beta_n\} \subset [0,1]$, $\sum\limits_{n=1}^{\infty} \alpha_n\beta_n < \infty$, $\alpha_n\beta_n L_i^2 < 1$ for all $n \geq 1$ and $i \in \{1, 2 \cdots N\}$ and $\sum\limits_{n=1}^{\infty} \alpha_n^2 < \infty$. Then $\{x_n\}$ converges strongly to a common fixed point of $T_1, T_2, \cdots, T_N$ if and only if $\liminf\limits_{n\to\infty} d(x_n, F) = 0$, where $d(x_n, F) = \inf\limits_{q\in F} \|x_n - q\|$.

Proof Using the same method as given Theorem 2.1, we can prove Theorem 2.2. This completes the proof.

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关于Lipschitzian伪压缩映射的合成隐迭代序列

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摘要: 本文研究了Lipschitzian伪压缩映射的合成隐迭代序列. 利用伪压缩映射等价不等式, 在Banach空间中, 得到了合成隐迭代序列强收敛的充分必要条件, 推广了一些相关的结果.

关键词: 伪压缩映射; 不动点; 合成隐迭代序列; Banach空间

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