THE EXTREMAL RANK SOLUTIONS OF THE MATRIX EQUATIONS $AX = B, XC = D$

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Abstract: In this paper, we considered the rank range of the solutions of a class of matrix equations. By applying the singular value decomposition of matrix and the properties of Frobenius matrix norm, we obtained the extremal rank and the solution expression of under rank constrained. Some special cases of these problems are considered, and some results are obtained.

Keywords: optimal control; extremal rank; SVD decomposition; Frobenius matrix norm

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1 Introduction

We first introduce some notations to be used. Let $C^{n \times m}$ denote the set of all $n \times m$ complex matrices; $R^{n \times m}$ denote the set of all $n \times m$ real matrices; $OR^{n \times n}$ be the set of all $n \times n$ orthogonal matrices. The symbols $A^T$, $A^+$, $A^-$, $R(A)$, $N(A)$ and $r(A)$ stand for the transpose, Moore-Penrose generalized inverse, any generalized inverse, range(column space), null space and rank of $A \in R^{n \times m}$, respectively. The symbols $E_A$ and $F_A$ stand for the two projectors $E_A = I - AA^+$ and $F_A = I - A^-A$ induced by $A$. The matrices $I$ and $0$, respectively, denote the identity and zero matrices of sizes implied by context. We use $\langle A, B \rangle = \text{trace}(B^T A)$ to define the inner product of matrices $A$ and $B$ in $R^{n \times m}$. Then $R^{n \times m}$ is a Hilbert inner product space. The norm of a matrix generated by the inner product is the Frobenius norm $\| \cdot \|$, that is $\|A\| = \sqrt{\langle A, A \rangle} = (\text{trace}(A^T A))^{\frac{1}{2}}$.

Researches on extreme ranks of solutions to linear matrix equations was actively ongoing for more than 30 years. For instance, Mitra [1] considered solutions with fixed ranks for the matrix equations $AX = B$ and $AXB = C$; Mitra [2] gave common solutions of minimal rank of the pair of matrix equations $AX = C, XB = D$; Uhlig [3] gave the maximal and minimal ranks of solutions of the equation $AX = B$; Mitra [4] examined common solutions of minimal rank of the pair of matrix equations $A_1X_1B_1 = C_1$ and $A_2X_2B_2 = C_2$. By applying the matrix rank method, recently, Tian [5] obtained the minimal rank of solutions to the matrix equation $AX = B X + Y C$. In 2003, Tian in [6, 7] investigated the extremal ranks solutions to the complex matrix equation $AXB = C$ and gave some applications. In 2006, Lin and Wang
in [8] studied the extreme ranks of solutions to the system of matrix equations \( A_1X = C_1, XB_2 = C_2, A_3XB_3 = C_3 \) over an arbitrary division ring, which was investigated in [9] and [10]. Recently, Xiao et al. considered the extreme ranks, i.e. maximal and minimal ranks to the equation \( AX = B \) (see, e.g. [11–15]).

In this paper, we consider the extreme rank solutions of the matrix equations

\[
AX = B, \; XC = D, \tag{1.1}
\]

where \( A \in R^{p \times m}, B \in R^{p \times n}, C \in R^{n \times q}, D \in R^{m \times q} \) are given matrices.

The paper is organized as follows. At first, we will introduce several lemmas which will be used in the latter sections. In Section 3, applying the matrix rank method, we will discuss the rank of the general solution to the matrix equations \( AX = B, XC = D \), where \( A \in R^{p \times m}, B \in R^{p \times n}, C \in R^{n \times q}, D \in R^{m \times q} \) are given matrices.

2 Some Lemmas

**Lemma 2.1** (see [6]) Let \( A, B, C, \) and \( D \) be \( m \times n, \; m \times k, \; l \times n, \; l \times k \) matrices, respectively. Then

\[
r \begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(C(I - A^{-1}A)), \tag{2.1}
\]

\[
r \begin{bmatrix} A & B \\ C & D \end{bmatrix} = r \begin{bmatrix} A \\ C \end{bmatrix} + r \begin{bmatrix} A & B \end{bmatrix} - r(A) + r(E_G(D - CA^{-1}B)F_H), \tag{2.2}
\]

where \( G = CF_A \) and \( H = E_AB \).

**Lemma 2.2** (see [16]) Given \( A \in R^{p \times m}, B \in R^{p \times n}, C \in R^{n \times q}, D \in R^{m \times q} \). Let the singular value decompositions of \( A \) be,

\[
A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T = U_1 \Sigma V_1^T, \tag{2.3}
\]

where \( U = (U_1, U_2) \in OR^{p \times p}, \; U_1 \in R^{p \times k}, \; V = (V_1, V_2) \in OR^{m \times m}, \; V_1 \in R^{m \times k}, \; k = r(A), \; \Sigma = \text{diag}(\sigma_1, \sigma_2, \cdots, \sigma_k), \; \sigma_1 \geq \cdots \geq \sigma_k > 0 \). Let the singular value decompositions of \( B \) be,

\[
C = P \begin{bmatrix} \Gamma & 0 \\ 0 & 0 \end{bmatrix} Q^T = P_1 \Gamma Q_1^T, \tag{2.4}
\]

where \( P = (P_1, P_2) \in OR^{n \times n}, \; P_1 \in R^{n \times t}, \; Q = (Q_1, Q_2) \in OR^{q \times q}, \; Q_1 \in R^{q \times t}, \; t = r(C), \; \Gamma = \text{diag}(\gamma_1, \gamma_2, \cdots, \gamma_t), \; \gamma_1 \geq \cdots \geq \gamma_t > 0 \). Then the matrix equations (1.1) have a solution in \( R^{m \times n} \) if and only if

\[
BC = AD, \quad AA^*B = B, \quad DC^*C = C. \tag{2.5}
\]

Moreover, its general solution can be expressed as

\[
X = DC^* + A^*B - A^*ADC^* + (I - A^*A)Z(I - CC^*), \forall Z \in R^{m \times n}. \tag{2.6}
\]
Lemma 2.3 Suppose that matrix equations (1.1) is consistent. Let the singular value decompositions of $A$ and $C$ given by (2.3) and (2.4), respectively. Denote by $X$ the solution of matrix equations (1.1). Then matrix $V^T XP$ can be partitioned into

$$
V^T XP = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix},
$$

(2.7)

where

$$
X_{11} = V_1^T X P_1 = \Sigma^{-1} U_1^T BP_1 \in R^{k \times t}, X_{12} = V_1^T X P_2 = \Sigma^{-1} U_1^T BP_2 \in R^{k \times (n-t)},
$$

$$
X_{21} = V_2^T X P_1 = V_2^T D Q_1 \Gamma^{-1} \in R^{(m-k) \times t}, X_{22} \in R^{(m-k) \times (n-t)}
$$

is arbitrary.

Proof By (2.6), $Z$ is arbitrary, we claim from (2.3), (2.4) and (2.7) that $X_{22}$ is arbitrary too. We omit the proof.

3 The Extremal Rank Solutions to (1.1)

Assume the matrix equations (1.1) has a solution $X \in R^{m \times n}$, and the general solution can be written as

$$
X = V \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} P^T, \quad \forall X_{22} \in R^{(m-k) \times (n-t)},
$$

(3.1)

where

$$
X_{11} = \Sigma^{-1} U_1^T BP_1 = V_1^T X P_1, X_{12} = \Sigma^{-1} U_1^T BP_2 = V_1^T X P_2,
$$

$$
X_{21} = V_2^T D Q_1 \Gamma^{-1} = V_2^T X P_1.
$$

Let $G_1 = X_{21} F_{X_{11}}, H_1 = E_{X_{11}}, X_{12}$. Assume the singular value decomposition of $G_1$ and $H_1^+$ be, respectively,

$$
G_1 = U_{G_1} \begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix} V_{G_1}^T = U_{11} \Sigma_1 V_{11}^T,
$$

(3.2)

where $U_{G_1} = (U_{11}, U_{12}) \in OR^{(m-k) \times (m-k)}, U_{11} \in R^{(m-k) \times k_1}, V_{G_1} = (V_{11}, V_{12}) \in OR^{k \times k}$,

$$
V_{11} \in R^{k \times k_1}, k_1 = r(G_1), \Sigma_1 = \text{diag}(\alpha_{11}, \alpha_{21}, \ldots, \alpha_{k_1}), \alpha_{11} \geq \cdots \geq \alpha_{k_1} > 0.
$$

$$
H_1^+ = P_{H_1^+} \begin{pmatrix} \Gamma_1 & 0 \\ 0 & 0 \end{pmatrix} Q_{H_1^+}^T = P_{11} \Gamma_1 Q_{11}^T,
$$

(3.3)

where $P_{H_1^+} = (P_{11}, P_{12}) \in OR^{(n-t) \times (n-t)}, P_{11} \in R^{(n-t) \times t_1}, Q_{H_1^+} = (Q_{11}, Q_{12}) \in OR^{k \times k}$,

$$
Q_{11} \in R^{k \times t_1}, t_1 = r(H_1^+), \Gamma_1 = \text{diag}(\beta_{11}, \beta_{21}, \ldots, \beta_{t_1}), \beta_{11} \geq \cdots \geq \beta_{t_1} > 0.
$$

Now we can establish the existence theorems as follows.
Theorem 3.1  Given $A \in \mathbb{R}^{p \times m}$, $B \in \mathbb{R}^{p \times n}$, $C \in \mathbb{R}^{n \times q}$, $D \in \mathbb{R}^{m \times q}$. the singular value decompositions of the matrices $A$, $C$ and $G_1$, $H_1^T$ are given by (2.3), (2.4) and (3.2), (3.3), respectively. Then equations (1.1) has a solution $X$ if and only if

$$BC = AD, \quad AA^+ B = B, \quad DC^+ C = C. \tag{3.4}$$

In this case, let $\Omega$ be the set of all solutions of equations (1.1), then the extreme ranks of $X$ are as follows:

1. The minimal rank of $X$ is

$$\min_{X \in \Omega} r(X) = r(B) + r(D) - r(BC). \tag{3.5}$$

The general expression of $A$ satisfying (3.5) is

$$X = X_0 + V_2 U_{11} U_{11}^T \tilde{Y} P_1 P_1^T P_2^T, \tag{3.6}$$

where $X_0 = DC^+ + A^+ B - A^+ ADC^+ + (I - AA^+) DC^+ (A^+ BCC^+)^+ A^+ B (I - CC^+)$, and $\tilde{Y} \in \mathbb{R}^{(m-k) \times (n-t)}$ is arbitrary matrix.

2. The maximal rank of $X$ is

$$\max_{X \in \Omega} r(X) = \min(m + r(B) - r(A), n + r(D) - r(C)). \tag{3.7}$$

The general expression of $X$ satisfying (3.7) is

$$X = X_0 + V_2 Y P_2^T, \tag{3.8}$$

where

$$X_0 = DC^+ + A^+ B - A^+ ADC^+ + (I - AA^+) DC^+ (A^+ BCC^+)^+ A^+ B (I - CC^+),$$

and the arbitrary matrix $Y \in \mathbb{R}^{(m-k) \times (n-t)}$ satisfies

$$r(E_{G_1} Y F_{H_1}) = r(BC) + \min(m + r(A) - r(D), n - r(B) - r(C)).$$

Proof  Suppose the matrix equation (1.1) has a solution $X$, then it follows from Lemma 2.2 that (3.4) hold. In this case, let $\Omega$ be the set of all solutions of equations (1.1). By (3.1),

$$r(X) = r \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}. \tag{3.9}$$

By Lemma 2.1, we have

$$r(X) = r \begin{bmatrix} X_{11} \\ X_{21} \end{bmatrix} + r [X_{11} \ X_{12}] - r(X_{11}) + r[E_{G_1} (X_{22} - X_{21} X_{11}^+ X_{12}) F_{H_1}], \tag{3.10}$$

where $G_1 = X_{21} F_{X_{11}}$, $H_1 = E_{X_{11}} X_{12}$. 


\[
\begin{align*}
   r \begin{bmatrix} X_{11} \\ X_{21} \end{bmatrix} &= r \begin{bmatrix} V_1^T X P_1 \\ V_2^T X P_1 \end{bmatrix} = \begin{bmatrix} V_1^T D Q_1 \Gamma^{-1} \\ V_2^T D Q_1 \Gamma^{-1} \end{bmatrix} = r(V^T D Q_1 \Gamma^{-1}) \\
   &= r(D Q_1) = r(D Q_1 Q^T_1) = r(D C^+ C) = r(D), \\
   r \begin{bmatrix} X_{11} \\ X_{12} \end{bmatrix} &= r \begin{bmatrix} \Sigma^{-1} U_1^T B P_1 & \Sigma^{-1} U_1^T B P_2 \end{bmatrix} = r(\Sigma^{-1} U_1^T B(P_1, P_2)) \\
   &= r(\Sigma^{-1} U_1^T B) = r(U_1 U_1^T B) = r(A A^+ B) = r(B), \\
   r(X_{11}) &= r(\Sigma^{-1} U_1^T B P_1) = r(U_1 U_1^T B P_1 \Gamma) = r(U_1 U_1^T B P_1 \Gamma Q_1^T) = r(A A^+ B C) = r(B C).
\end{align*}
\]

By (3.10),
\[
\min_{X \in \Omega} r(X) = r(B) + r(D) - r(B C) + \min_{X_{22}} [E_{G_1}(X_{22} - X_{21} X_{11}^+ X_{12}) F_{H_1}]
\]
\[
= r(B) + r(D) - r(B C).
\]

Then (3.4) hold. By Lemma 2.2, The general expression of \(X\) satisfying (3.5) can be expressed as
\[
X = D C^+ + A^+ B - A^+ A D C^+ + V_2 X_{21} X_{11}^+ X_{12} P_2^T + V_2 Y P_2^T,
\] (3.11)
where \(Y \in R^{(m-k) \times (n-t)}\) satisfies \(E_{G_1} Y F_{H_1} = 0\).

By (3.1),
\[
X_{11} = \Sigma^{-1} U_1^T B P_1 \in R^{k \times t}, X_{12} = \Sigma^{-1} U_1^T B P_2 \in R^{k \times (n-t)}, X_{21} = V_2^T D Q_1 \Gamma^{-1} \in R^{(m-k) \times t}.
\]

By (2.3), (2.4), \(A^+ = V_1 \Sigma^{-1} U_1^T, \quad C C^+ = P_1 P_1^T\). Then
\[
V_1 X_{11} P_1^T = A^+ B C C^+, \quad P_1 X_{11} V_1^T = (A^+ B C C^+)^+, \quad X_{11}^+ = P_1^T (A^+ B C C^+)^+ V_1.
\]

Thus we obtain
\[
V_2 X_{21} X_{11}^+ X_{12} P_2^T = \begin{bmatrix} V_2^T D Q_1 \Gamma^{-1} P_1^T (A^+ B C C^+)^+ V_1 \Sigma^{-1} U_1^T B P_2 P_2^T \\
   = (I - A A^+) D C^+ (A^+ B C C^+)^+ A^+ B (I - C C^+).
\end{bmatrix}
\]

By (3.2), (3.3),
\[
G_1 G_1^+ = U_{11} U_{11}^T, \quad E_{G_1} = I - U_{11} U_{11}^T = U_{12} U_{12}^T,
\]
\[
H_1 H_1^+ = P_1 P_1^T, \quad F_{H_1} = I - P_1 P_1^T = P_{12} P_{12}^T.
\]

Thus \(E_{G_1} Y F_{H_1} = 0, \text{ i.e. } U_{12} U_{12}^T Y P_{12} P_{12}^T = 0\), we have
\[
Y = U_{11} U_{11}^T \tilde{Y} P_{11} P_{11}^T,
\] (3.13)
where \(\tilde{Y} \in R^{(m-k) \times (n-t)}\) is arbitrary.

Taking (3.12), (3.13) into (3.11) yields (3.6).
(2) By (3.10),
\[
\max_{X \in \Omega} r(X) = r(B) + r(D) - r(BC) + \max_{X_{22}} r[E_{G_1}(X_{22} - X_{11}^T X_{12})F_{H_1}]
\]
\[
= r(B) + r(D) - r(BC) + \min(r(E_{G_1}), r(F_{H_1})).
\]

Since \(E_{G_1}\) and \(F_{H_1}\) are idempotent matrices, we have
\[
r(E_{G_1}) = \text{trace}(E_{G_1}) = m - k - r(G_1 G_1^T) = m - k - r(G_1)
\]
\[
= m - k - r(X_{21}(I - X_{11}^T X_{11})) = m - k - r\left[\begin{array}{c} X_{11} \\ X_{21} \end{array}\right] + r(X_{11})
\]
\[
r(F_{H_1}) = \text{trace}(F_{H_1}) = n - t - r(H_1 H_1^T) = n - t - r(H_1)
\]
\[
= n - t - r(I - X_{11} X_{11}^T)X_{12}) = n - t - r[X_{11} X_{12}] + r(X_{11})
\]
\[
= n - t - r(B) + r(BC) = n + r(BC) - r(B) - r(C).
\]

Then the maximal rank of the matrix equations (1.1) is
\[
\max_{X \in \Omega} r(X) = \min(m + r(BC) - r(A) - r(D), n + r(BC) - r(B) - r(C)) + r(B) + r(D) - r(BC) = \min(m + r(B) - r(A), n + r(D) - r(C)).
\]

By Lemma 2.2, The general expression of \(X\) satisfying (3.7) can be expressed as
\[
X = X_0 + V_2 Y P_2^T,
\]
where \(X_0 = D C^+ + A^+ B - A^+ A D C^+ + (I - A A^+) D C^+ (A^+ B C C^+) A^+ B (I - C C^+)\), and the arbitrary matrix \(Y \in R^{(m-k) \times (n-t)}\) satisfies
\[
r(E_{G_1}, Y F_{H_1}) = r(BC) + \min(m - r(A) - r(D), n - r(B) - r(C)).
\]

The proof is completed.

The result in (3.5) implies Theorem 3 in (see [2]) as a corollary.

**Corollary** Assume \(r(B) \leq r(D)\), and matrix equations (1.1) is consistent. Then the matrix equations (1.1) have solution with rank of \(r(D)\) if and only if \(r(BC) = r(B)\).

**References**


矩阵方程 $AX = B$, $XC = D$ 的定秩解

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摘要: 本文研究了一类矩阵方程组解的秩的范围, 利用矩阵的奇异值分解以及Frobenius范数的特征, 得到了解的秩取值以及解的通式, 并就这些问题的特殊情况进行了讨论，得到了一些结果。

关键词: 最优控制; 极值秩; 奇异值分解; Frobenius范数